Introduction to dynamo theory

Lecture 1

17/02/2024

Plan for this lecture (2 hrs)

1) Observations of magnetic fields in space

2) Modelling magnetized fluids with magnetohydrodynamics

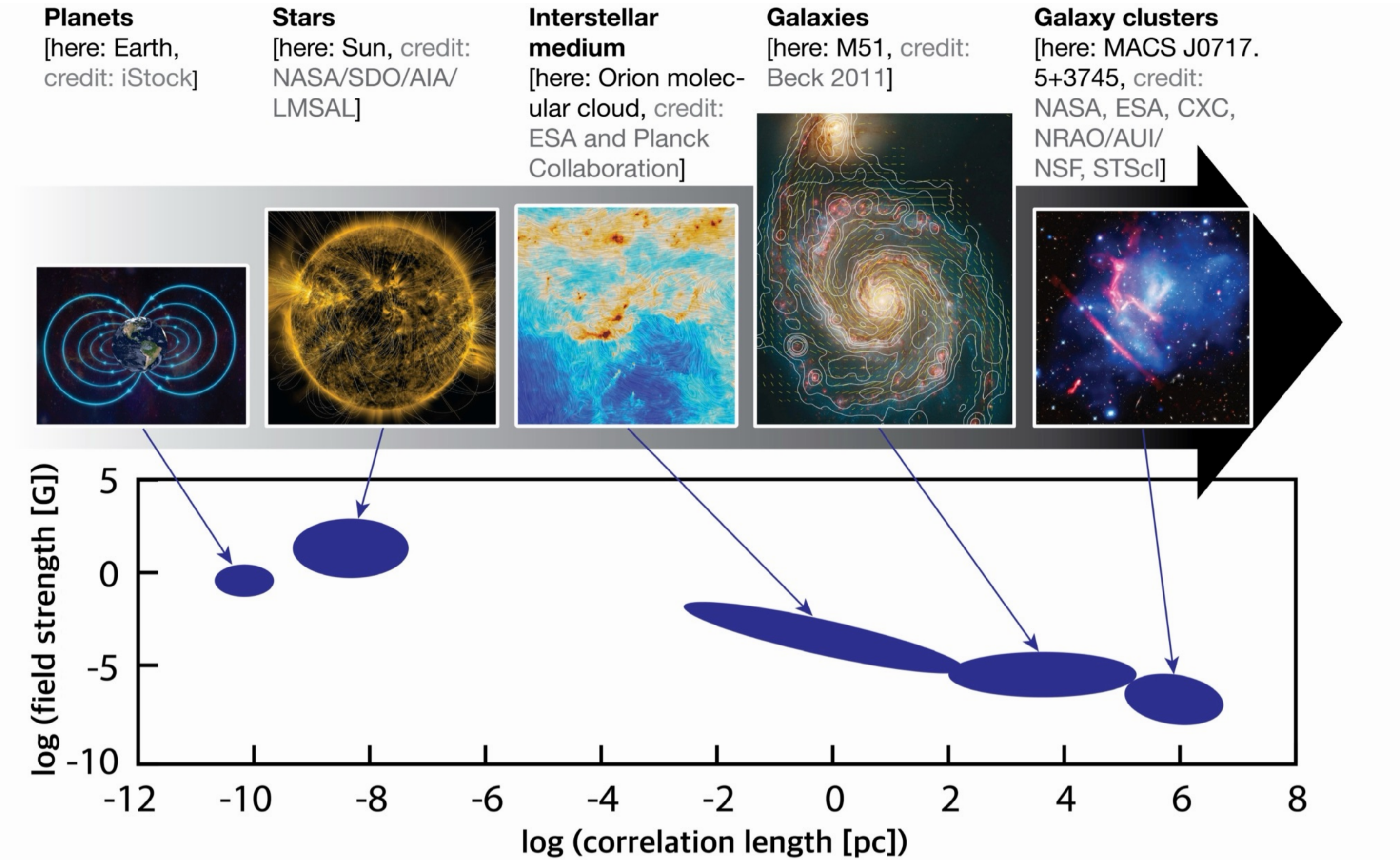
3) Turbulent dynamos – an overview

4) Mean-field dynamo

5) Small-scale dynamo

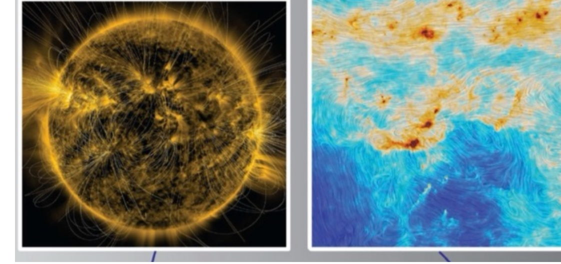
1. Observational facts

1.1 Field strengths and correlation lengths



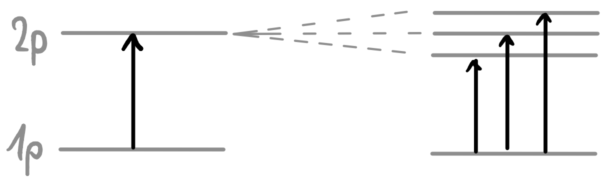
Technique:

Zeeman splitting of spectral lines

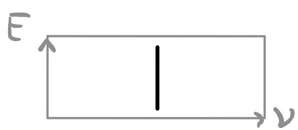


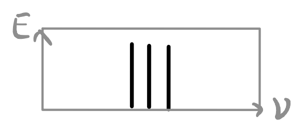
no B-field: with B-field:

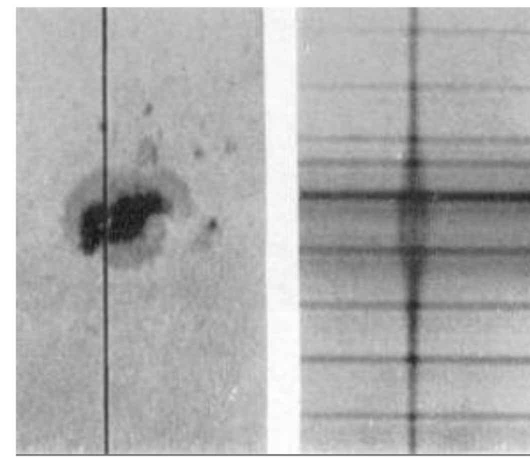
Transition between atomic energy levels:



Resulting spectra:



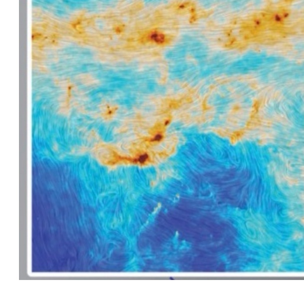




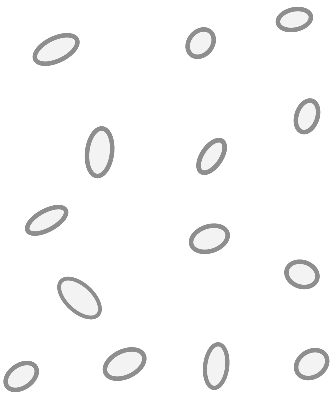
[Hale 1908]

Technique:

Dust grain alignment

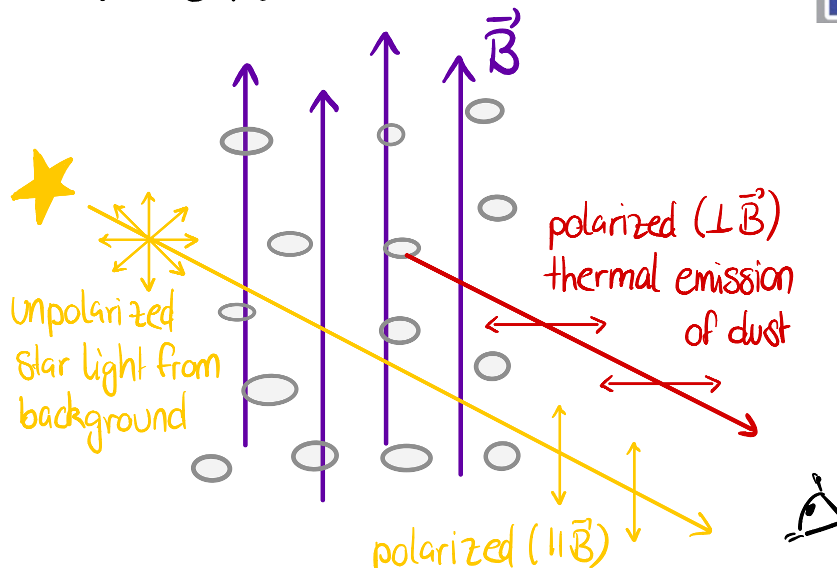


no B-field:



=> random orientation of dust grains

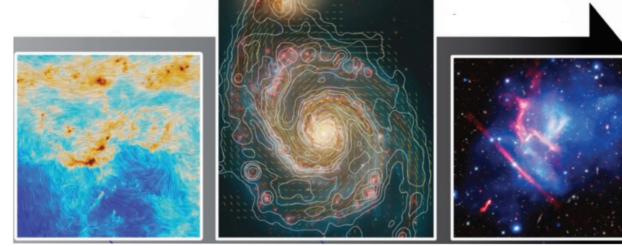
with B-field:

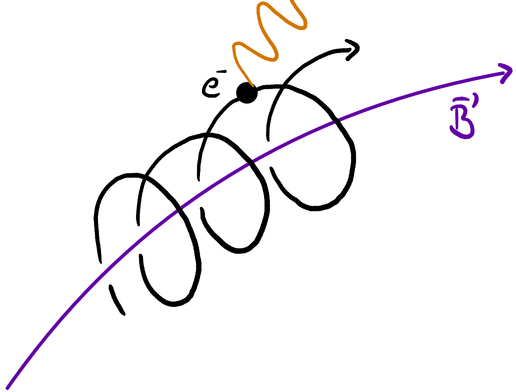


=> alignment with $\vec{B}$ leads to polarization

Technique:

Synchrotron emission of cosmic ray electrons

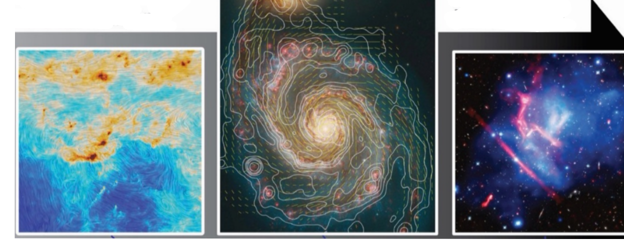


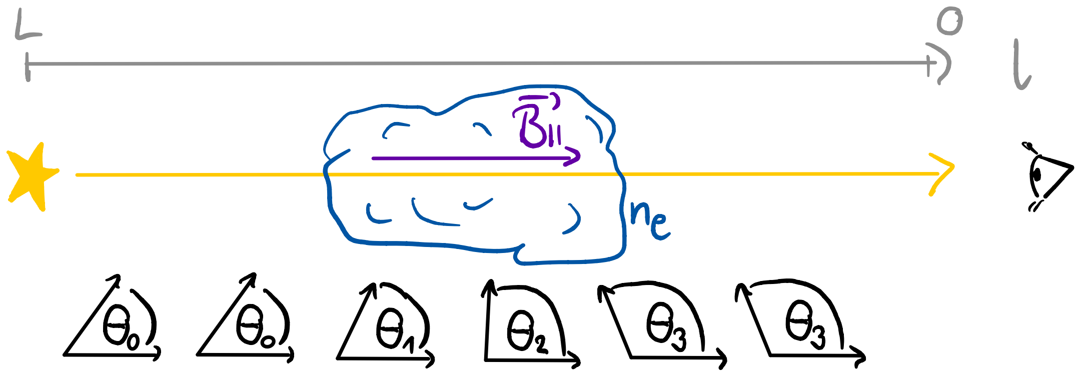


$$\boxed{L_{synchrotron} = \int_1^\infty \underbrace{L_e(B, \gamma)}_{\text{emission of individual } e^- \text{ (depends on } \gamma)} \cdot \underbrace{N_e(\gamma)}_{\text{cosmic ray distribution}} \, d\gamma}$$

Technique :

Faraday rotation



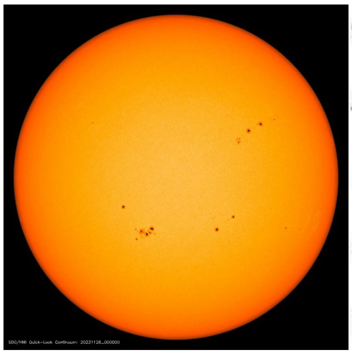


$$\boxed{\frac{\partial \Delta\theta}{\partial \omega} = - \frac{4\pi e^3}{m_e^2 c^2} \frac{1}{\omega^3} \int_0^L n_e B_{\parallel} \, dl}$$

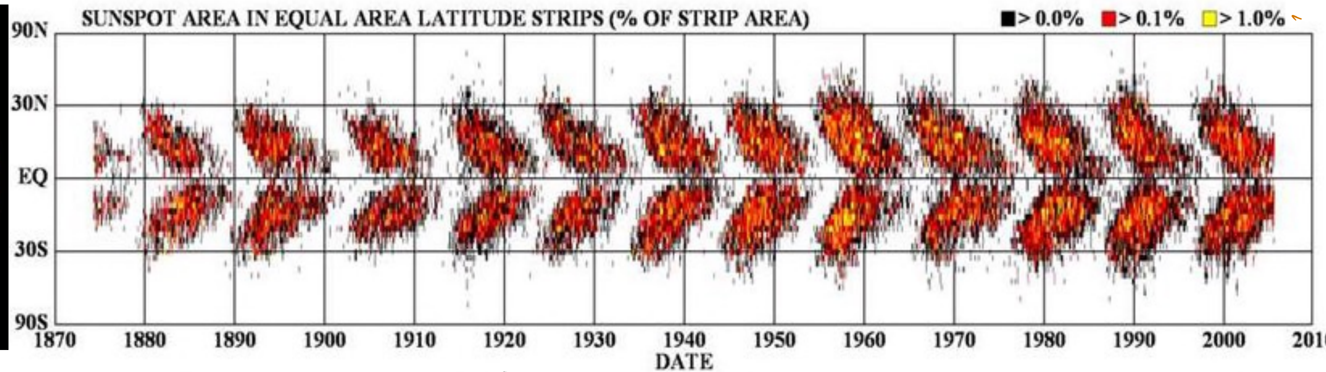
1.2 Time dependence of magnetic fields ?

Solar cycle :

22 yrs



[HMI, SDO, NASA]

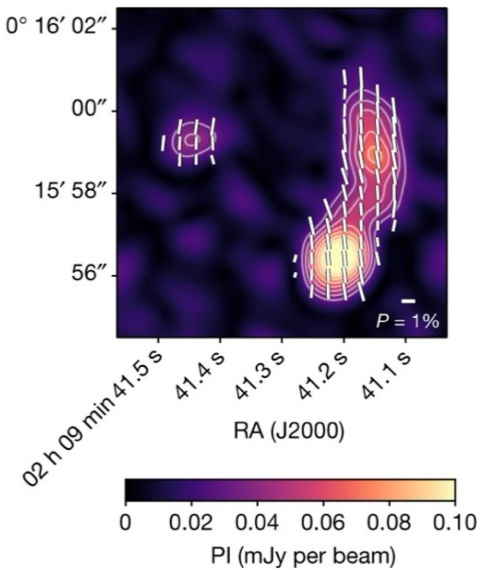


[Royal Observatory, Greenwich]

"Butterfly diagram"

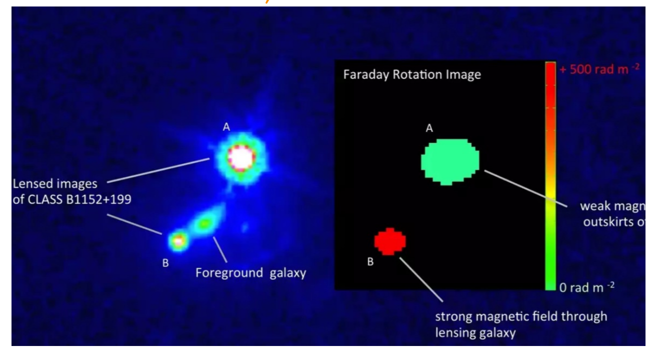
Observations of magnetic fields in highly redshifted galaxies

$B \lesssim 500\,\mu G$



[Geach et al. 2023]

$B \approx \mu G$



[Mao et al. 2017]

$B \approx 10\,\mu G$

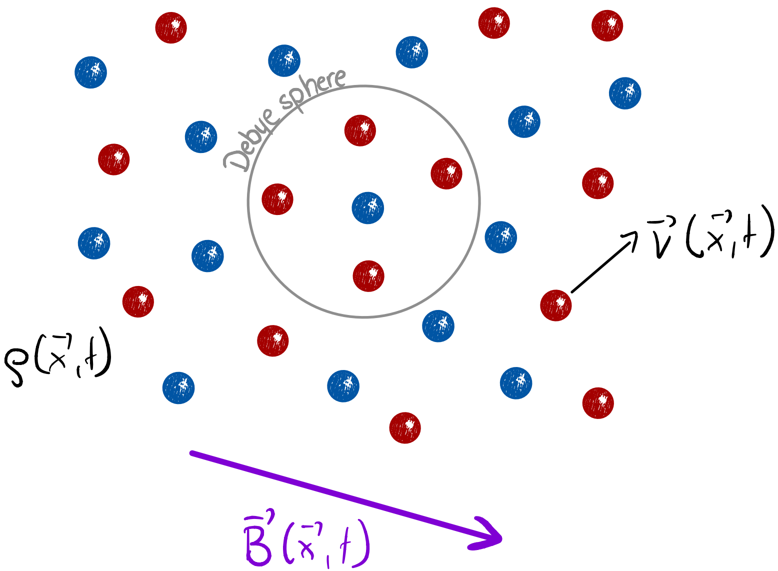
[SOFIA]

Big Bang

$z = \infty$	2.6	0.44	0
$t = 0$	2.5 Gyrs	9.0 Gyrs	13.6 Gyrs

2. Theoretical background

2.1 Magnetohydrodynamics $\hat{=}$ effective model for a magnetized fluid



The effective model (= MHD) is possible if

- quasi-neutrality applies (length > Debye length),
- the plasma is collisional (length > mfp),
- timescales are long (time > $\nu_{c,\text{ion}}^{-1}$), and
- flows are non-relativistic ($v \ll c$).

Full set of equations for magnetohydrodynamics (MHD):

Mass conservation:

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0} \qquad (*)$$

Momentum conservation:

$$\boxed{\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{1}{\rho}\nabla\left(p + \frac{B^2}{8\pi}\right) + \frac{1}{4\pi\rho}(\vec{B}\cdot\nabla)\vec{B} + \frac{1}{m}\vec{F} + \nu \nabla^2 \vec{v}} \qquad (**)$$

Induction equation:

$$\boxed{\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) + \eta \nabla^2 \vec{B}} \qquad (***)$$

Equation of state:

$$\boxed{\rho = \rho(p)} \qquad (****)$$

\# of variables: $2\ (\rho, p \text{ or } T) + 3\ (\vec{v}) + 3\ (\vec{B})$

\# of equations: $1\ (*) + 3\ (**) + 3\ (***) + 1\ (****)$

2.2 The induction equation

$\Rightarrow$ Evolution equation of magnetic field.

Derivation

Ampere's law (without displacement current):

$$\nabla \times \vec{B} = \frac{4\pi}{c} \vec{j}$$

| Ohm's law: $\boxed{\vec{j} = \underset{\substack{\uparrow \\ \text{electrical conductivity}}}{\sigma} \left(\vec{E} + \frac{1}{c} \vec{v} \times \vec{B} \right)}$

$$\Rightarrow \nabla \times \vec{B} = \frac{4\pi}{c} \sigma \left(\vec{E} + \frac{1}{c} \vec{v} \times \vec{B} \right)$$

| $\nabla \times$

$$\Rightarrow \nabla \times (\nabla \times \vec{B}) = \frac{4\pi\sigma}{c} \left(\nabla \times \vec{E} + \frac{1}{c} \nabla \times (\vec{v} \times \vec{B}) \right)$$

| Faraday's law: $\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$

& $\nabla \times (\nabla \times \vec{B}) = \nabla (\nabla \cdot \vec{B}) - \nabla^2 \vec{B}$

$$\Rightarrow \boxed{\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) + \underbrace{\frac{c^2}{4\pi\sigma}}_{\equiv \eta} \nabla^2 \vec{B}}$$

"magnetic diffusivity"

Different limits

Write induction equation with dimensionless parameters:

$$\vec{B} \to B\,\tilde{\vec{B}}, \quad \vec{\nabla} \to \frac{1}{L}\tilde{\vec{\nabla}}, \quad \vec{v} \to V\,\tilde{\vec{v}}, \quad t \to \frac{L}{V}\tilde{t}$$

$$\Rightarrow \frac{\partial \tilde{\vec{B}}}{\partial \tilde{t}} \frac{\cancel{B}}{L} V = \frac{1}{L} V \cancel{B} \left[\tilde{\vec{\nabla}} \times \left(\tilde{\vec{v}} \times \tilde{\vec{B}}\right)\right] + \eta \frac{1}{L^2} \cancel{B}\, \tilde{\vec{\nabla}}^2 \tilde{\vec{B}}$$

$$\Rightarrow \frac{\partial \tilde{\vec{B}}}{\partial \tilde{t}} = \tilde{\vec{\nabla}} \times \left(\tilde{\vec{v}} \times \tilde{\vec{B}}\right) + \underbrace{\frac{\eta}{VL}}_{\equiv \frac{1}{Re_M}} \tilde{\vec{\nabla}}^2 \tilde{\vec{B}}$$

$$\Rightarrow \boxed{Re_M = \frac{V \cdot L}{\eta} = \begin{cases} \ll 1 & \Rightarrow \text{diffusion limit} \\ \gg 1 & \Rightarrow \text{advection limit} \end{cases}} \quad \begin{matrix} \Rightarrow \text{lab} \\ \Rightarrow \text{astrophysics} \end{matrix}$$

"magnetic Reynolds number"

Magnetic flux freezing

Consider $Re_M \gg 1$:

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B})$$

Magnetic flux :

$$\psi = \int_S \vec{B}\, d\vec{S}$$

can change in time by either :

$$\left(\frac{\partial \psi}{\partial t}\right)_1 = \int_S \frac{\partial \vec{B}}{\partial t}\, d\vec{S} = -c \int_S \nabla \times \vec{E}\, d\vec{S} \qquad \Big|\ \vec{B} \text{ changes}$$

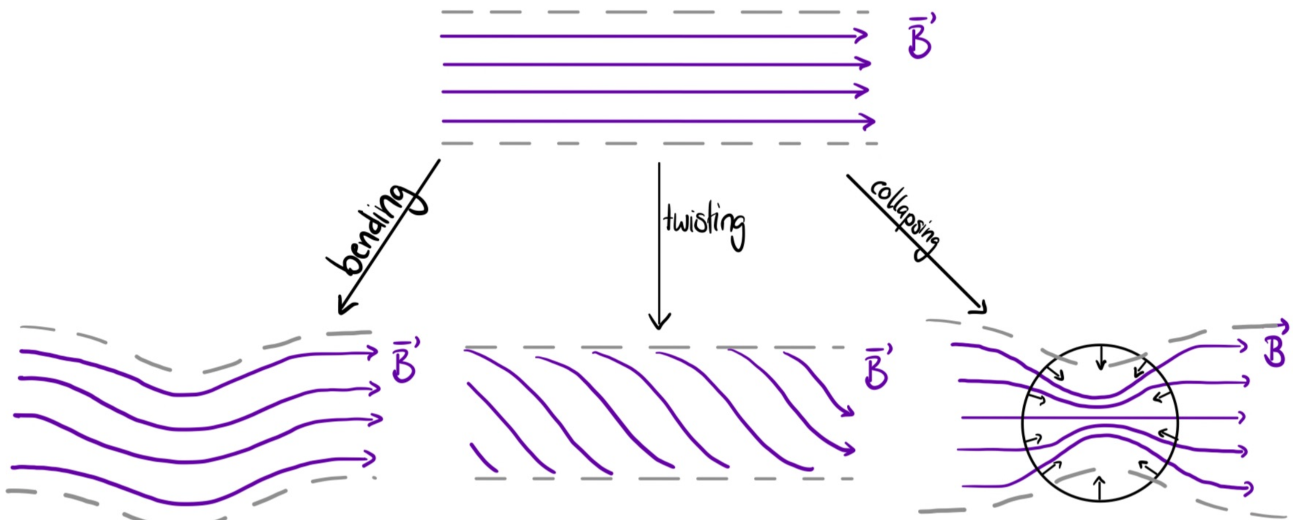
$$\left(\frac{\partial \psi}{\partial t}\right)_2 = \oint_C \vec{B} \cdot \vec{v} \times d\vec{l} = \oint_C \vec{B} \times \vec{v}\, d\vec{l} = \int_S \nabla \times (\vec{B} \times \vec{v})\, d\vec{S} \qquad \Big|\ \text{surface changes}$$

$$\Rightarrow \frac{\partial \Psi}{\partial t} = -\int_S \vec{\nabla} \times \underbrace{(\vec{v} \times \vec{B} + c\vec{E})}_{= \frac{c}{\sigma}\vec{j}} \, d\vec{S}$$

$\Rightarrow$ In ideal MHD ($\sigma \to \infty$):

$$\boxed{\frac{\partial \Psi}{\partial t} = 0}$$

"Alfvén's flux freezing theorem"



2.3 (Hydrodynamic) turbulence

Navier-Stokes equation in dimensionless form

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = \frac{1}{m}\vec{F} - \frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{v}$$

$$\Big|\ \vec{v} \to \tilde{\vec{v}}\, V,\quad t \to \tilde{t}\,\frac{L}{V},\quad \vec{\nabla} \to \tilde{\vec{\nabla}}\,\frac{1}{L},\quad p \to \tilde{p}\,\rho V^2,\quad \vec{F} \to \tilde{\vec{F}}\,\frac{V^2}{L}$$

$$\Rightarrow \frac{\partial \tilde{\vec{v}}}{\partial \tilde{t}}\frac{V^2}{L} + (\tilde{\vec{v}}\cdot\tilde{\vec{\nabla}})\tilde{\vec{v}}\,\frac{V^2}{L} = \frac{1}{m}\tilde{\vec{F}}\,\frac{V^2}{L} - \frac{1}{\rho}\tilde{\vec{\nabla}}\tilde{p}\,\frac{\rho V^2}{L} + \nu\,\tilde{\vec{\nabla}}^2\tilde{\vec{v}}\,\frac{V}{L^2}$$

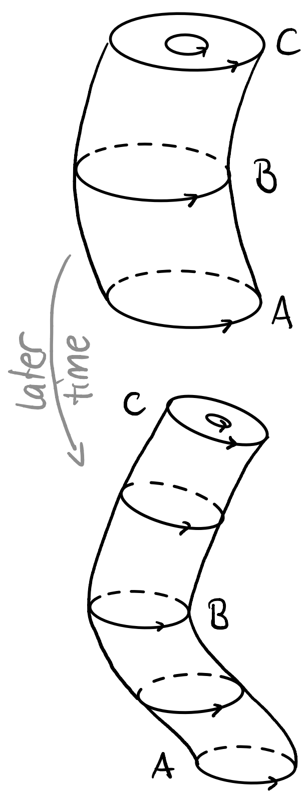
$$\Rightarrow \frac{\partial \tilde{\vec{v}}}{\partial \tilde{t}} + (\tilde{\vec{v}}\cdot\tilde{\vec{\nabla}})\tilde{\vec{v}} = \frac{1}{m}\tilde{\vec{F}} - \tilde{\vec{\nabla}}\tilde{p} + \underbrace{\frac{\nu}{VL}}_{=:\,Re^{-1}}\tilde{\vec{\nabla}}^2\tilde{\vec{v}}$$

$$\boxed{Re = \frac{V\cdot L}{\nu} \approx \frac{\vec{v}\cdot\vec{\nabla}\,\vec{v}}{\nu\,\vec{\nabla}^2\vec{v}} = \frac{\text{inertial forces}}{\text{viscous forces}} = \begin{cases} \ll 1 \ldots 10^{2-3} & \Rightarrow \text{laminar} \\ \gg 1 & \Rightarrow \text{turbulent} \end{cases}}$$

hydrodynamic Reynolds number

Phenomenological discription of turbulence Kolmogorov (1941)

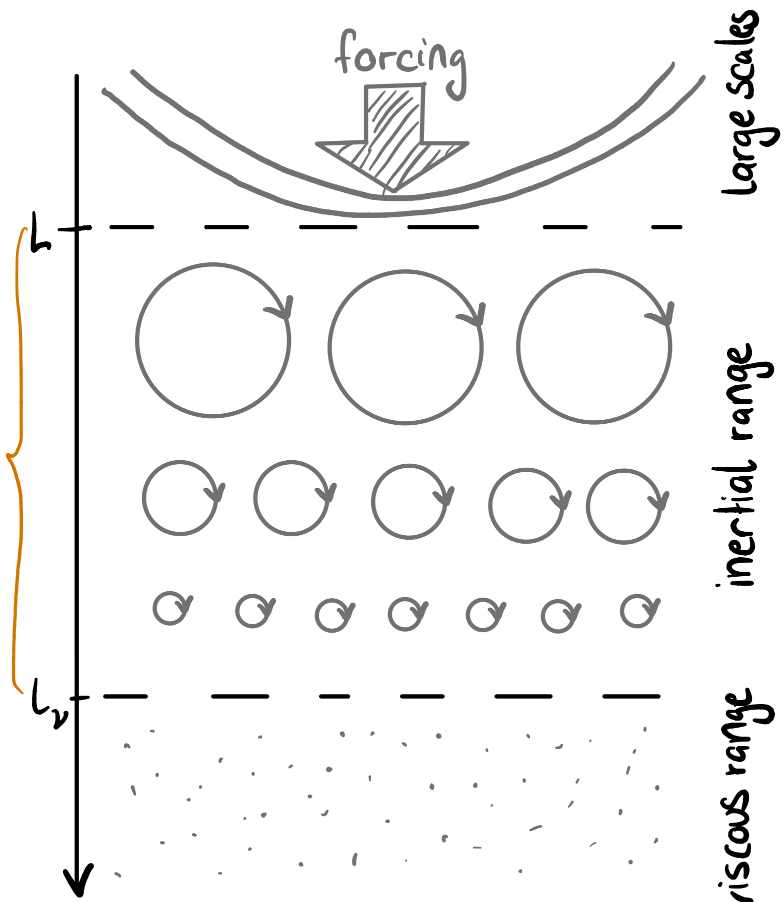
Basic idea: energy cascade



Same energy transfer rate

$$\varepsilon \approx \frac{v^3}{l}$$

on all scales



$$Re = \frac{V \cdot L}{\nu}$$

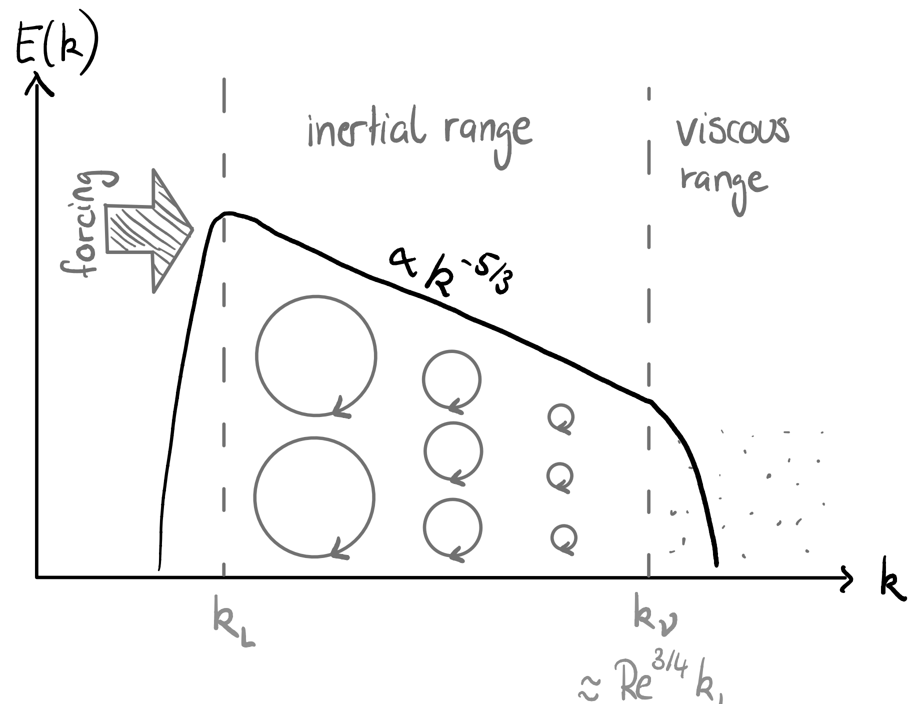
$$Re_l = \frac{v_l\, l}{\nu}$$

$$Re_\nu = \frac{v_\nu l_\nu}{\nu} \approx 1$$

$\Rightarrow$ kinetic energy density on $k \approx E(k)\,k = \frac{1}{2} v(l)^2 \approx \frac{1}{2} (\varepsilon l)^{2/3} = \frac{1}{2} \varepsilon^{2/3} k^{-2/3}$

$v = (\varepsilon l)^{1/3}$

$\Rightarrow \boxed{E(k) \propto k^{-5/3}}$ "$\frac{5}{3}$th law"



3. Dynamos – An overview

3.1 Why we need dynamos

Reason i) Need to sustain observed B-fields

How long does a magnetic field survive without sustaining mechanisms?

$$\frac{\partial \vec{B}}{\partial t} = \eta \nabla^2 \vec{B} \qquad \Big| \ \frac{\partial}{\partial t} \to \frac{1}{\tau_d}, \ \nabla \to \frac{1}{L}$$

$$\Rightarrow \frac{\vec{B}}{\tau_d} \approx \eta \frac{1}{L^2} \vec{B} \qquad \Big| \ \eta = \frac{c^2}{4\pi\sigma}$$

$$\Rightarrow \tau_d \approx \frac{4\pi\sigma L^2}{c^2} \qquad \Big| \ \sigma = 10^7 \, T^{3/2} \text{ e.s.u.} \quad \text{[Spitzer conductivity]}$$

Object	L [cm]	σ [e.s.u.]	τ_d [yrs]
Earth	$3 \cdot 10^8$	10^{16}	$3 \cdot 10^5$
Sun	$5 \cdot 10^{10}$	10^{17}	10^{11}
Galaxy	$3 \cdot 10^{20}$	10^{10}	$3 \cdot 10^{23}$

⇒ Need mechanism to sustain $\vec{B}$.

⇒ Does not explain reversal every 11 yrs.

⇒ Why is $\vec{B}$ not wound up in differential rotation?

Reason ii) Extremely weak seed magnetic fields

The induction equation is linear in $\vec{B}$:

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) - \eta \nabla^2 \vec{B}$$

$\Rightarrow$ If initially $\vec{B} = 0$, no magnetic field will be generated.

$\Rightarrow$ Seed magnetic fields are needed.

<u>Cosmological seed fields</u> (generation before recombination):

- Inflation scenarios ($t \lesssim 10^{-32}$ s): $B_0 \approx 10^{-65} - 10^{-9}$ G
  - $\rightarrow$ Fluctuations of (hyper)magnetic field are increased.
- Cosmological phase transitions ($t_{EW} \approx 10^{-11}$ s, $t_{QCD} \approx 10^{-6}$ s): $B_0 \approx 10^{-29} - 10^{-20}$ G
  - $\rightarrow$ Battery processes in non-equilibrium states.

<u>Astrophysical seed fields</u> (generation in late Universe)

→ Biermann battery resulting from two-fluid effects.

Consider the generalized Ohm's law:

$$\Rightarrow \vec{j} = \sigma \left[\vec{E} + \frac{1}{c} \vec{v} \times \vec{B} + \underset{\substack{\uparrow \\ \text{two fluid effect}}}{\frac{1}{ne}} \nabla p_e \right]$$

Derivation of induction equation:

$$\Rightarrow \frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) - \eta \nabla \times (\nabla \times \vec{B}) + \underbrace{\frac{c k_B}{e} \nabla T \times \left(\frac{\nabla n}{n} \right)}_{\text{Biermann battery}}$$

Estimate of field generated by Biermann:

$$\frac{\partial B}{\partial t} \approx \frac{c k_B}{e} \nabla T \times \left(\frac{\nabla n}{n}\right)$$

$$\Rightarrow \frac{B}{\tau_{ff}} \approx \frac{c k_B}{e} \frac{1}{L_J^2} T$$

$$\Rightarrow B \approx \frac{c \cancel{k_B}}{e} \frac{m^2 n G}{\cancel{k_B} \cancel{T}} \cancel{T} \frac{1}{(G m n)^{1/2}}$$

$$= \frac{c}{e} m^{3/2} n^{1/2} G^{1/2}$$

$1.67 \cdot 10^{-24} g$ $\quad 1\ cm^{-3}$ | value in interstellar medium

$$\approx 3 \cdot 10^{-20} G$$

$\ll 10^{-5} G$ | observed in galaxies today

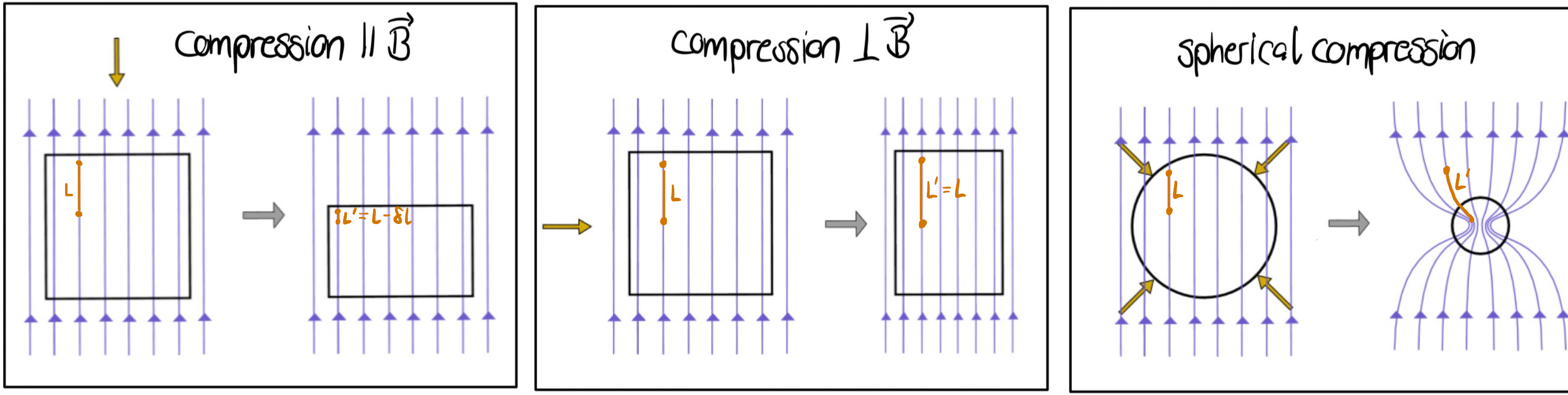
characteristic length of system:

Jeans length $= L_J \approx \frac{c_s}{(\rho G)^{1/2}} = \left(\frac{k_B T}{m^2 n G}\right)^{1/2}$

characteristic time scale of e^- movement:

free-fall time $= \tau_{ff} \approx \frac{1}{(G m n)^{1/2}}$

Reason iii) Inefficient gravitational compression of fields



Compression $\parallel \vec{B}$:

$$\delta l \propto \frac{1}{\rho}$$

$$\Rightarrow \frac{B}{\rho} \propto \frac{1}{\rho}$$

$$\Rightarrow B = \text{const}$$

Compression $\perp \vec{B}$:

$$\delta l = \text{const}$$

$$\Rightarrow \frac{B}{\rho} = \text{const}$$

$$\Rightarrow B \propto \rho$$

Spherical compression:

$$\delta l^{-3} \propto \rho$$

$$\Rightarrow \frac{B}{\rho} \propto \rho^{-1/3}$$

$$\Rightarrow B \propto \rho^{2/3}$$

Compressible ideal fluid:

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot(\rho\vec{v}) = -\rho\,\nabla\cdot\vec{v} - (\vec{v}\cdot\nabla)\rho \quad\Rightarrow\quad \frac{d\rho}{dt} = -\rho(\nabla\cdot\vec{v}) \qquad (*)$$

$$\frac{\partial\vec{B}}{\partial t} = \nabla\times(\vec{v}\times\vec{B}) = \cancel{\vec{v}(\nabla\cdot\vec{B})} - \vec{B}(\nabla\cdot\vec{v}) + (\vec{B}\cdot\nabla)\vec{v} - (\vec{v}\cdot\nabla)\vec{B}$$

$$\Rightarrow\quad \frac{d\vec{B}}{dt} = (\vec{B}\cdot\nabla)\vec{v} - \vec{B}(\nabla\cdot\vec{v}) \qquad (**)$$

Combine $(*)$ and $(**)$:

$$\Rightarrow\quad \frac{d\vec{B}}{dt} = (\vec{B}\cdot\nabla)\vec{v} + \frac{1}{\rho}\frac{d\rho}{dt}\vec{B} \qquad \Big|\cdot\frac{1}{\rho}$$

$$\Rightarrow\quad \frac{1}{\rho}\frac{d\vec{B}}{dt} - \frac{1}{\rho^2}\frac{d\rho}{dt}\vec{B} = \frac{1}{\rho}(\vec{B}\cdot\nabla)\vec{v}$$

$$\Rightarrow\quad \frac{d}{dt}\left(\frac{\vec{B}}{\rho}\right) = \left(\frac{\vec{B}}{\rho}\cdot\nabla\right)\vec{v} \qquad (***)$$

Compare with two fluid elements connected by field line $\vec{l}$:

$$\frac{d\vec{l}}{dt} = (\vec{v}_2 - \vec{v}_1)_l \rightarrow \frac{d\,\delta\vec{l}}{dt} = (\delta\vec{l} \cdot \nabla)\vec{v} \qquad (****)$$

Compare (***) and (****):

$$\frac{\vec{B}}{\rho} \propto \delta\vec{l}.$$

[see application to different compression scenarios above]

<u>Example</u>: Collapse of intergalactic cloud to galaxy

$$\rho_0 = m \cdot n_0 \approx m \cdot 10^{-6}\,\mathrm{cm}^{-3} \longrightarrow \rho_1 = m \cdot n_1 \approx m \cdot 1\,\mathrm{cm}^{-3}$$

$$B_0 = 10^{-20}\,G \xrightarrow{B \propto \rho^{2/3}} B_1 = B_0 \cdot \left(\frac{\rho_1}{\rho_0}\right)^{2/3} \approx B_0 \left(\frac{n_1}{n_0}\right)^{2/3} = B_0 \cdot 10^4 = 10^{-16}\,G$$

$$\ll 10^{-5}\,G$$