Les Houches

Physics of Star Formation

February 12-23, 2024

brief theoretical introduction

— basic equations —

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— lecture 1 —

more (useful) equations

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RECALL:

* gravitational MHD equations are conservation laws:

$$\frac{\partial}{\partial t}\left[\text{quantity}\right] + \vec{\nabla}\left\{\text{flux of quantity}\right\} = 0$$

* conserved quantities are mass, momentum, energy

\ continuity equation: $\boxed{\partial_t \rho + \partial_j \rho v_j = 0}$

↳ $$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + (\vec{v}\cdot\vec{\nabla})\rho = -\rho\,\vec{\nabla}\cdot\vec{v}$$

\ Navier-Stokes equation: $\boxed{\partial_t \rho v_i + \partial_j T_{ij} = 0}$

↳ $$\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla}\left(P + \frac{\vec{B}^2}{8\pi}\right) + \frac{1}{4\pi\rho}(\vec{B}\cdot\vec{\nabla})\vec{B} - \vec{\nabla}\Phi$$

\ energy equation: $\boxed{dQ = dU + dW}$ 1^{st} law of TD

↳ $$\frac{de}{dt} = \frac{\partial e}{\partial t} + (\vec{v}\cdot\vec{\nabla})e = T\frac{ds}{dt} + \frac{P}{\rho^2}\frac{d\rho}{dt}$$

Note for completeness: full energy equation for MHD

$$\boxed{\partial_t\left(\frac{1}{2}\rho\vec{v}^2 + h + \frac{\vec{B}^2}{8\pi}\right) + \partial_i q_i = 0}$$

with enthalpy = heat function
$h = \frac{1}{\rho}(\epsilon + P)$

and energy flux $\vec{q}$

$$q_i = \rho\left(\frac{\vec{v}^2}{2} + h\right)v_i - \underbrace{\kappa\,\partial_i T}_{\text{temperature gradients}} - \underbrace{\sigma_{ij}v_j}_{\text{internal stress}} + \underbrace{\frac{c}{4\pi}\epsilon_{ijk}E_j B_k}_{\text{Poynting vector}}$$

for self-gravity: add $\rho\phi$ to energy density and $\rho\phi v_i$ to flux

* needs closure equations:

- magnetic field: $$\frac{\partial\vec{B}}{\partial t} = \frac{c^2}{4\pi\sigma}\vec{\nabla}^2 B + \vec{\nabla}\times(\vec{v}\times\vec{B})$$ induction equation

- self-gravity: $$\vec{\nabla}^2\Phi = 4\pi G\rho$$ Poisson equation

- hydrodynamics: $$P = P(\rho, T)$$ equation of state

* more on equation of state:

- ideal gas EOS: $\boxed{P = \frac{k}{\mu m_p} \cdot \rho T}$

with k = Boltzmann constant $= 1{,}38 \cdot 10^{-16}$ erg / K

m_p = proton mass $= 1{,}67 \cdot 10^{-24}$ g

μ = mean molecular weight (depends on medium)

- barotropic EOS $\boxed{P = P(\rho)}$

- note: both EOS descriptions can be valid; this implies an implicit relation between ρ and T

- polytropic EOS $\boxed{P = K \rho^{\gamma}}$

γ = polytropic exponent $= \frac{\text{specific heat at } P = \text{const.}}{\text{specific heat at } V = \text{const.}} = \frac{c_p}{c_V}$

- recall that for monoatomic gases (only translational degrees of freedom):

$$\boxed{c_p = c_v + k}$$ per gas particle

we have $c_p = \frac{5}{2} k$ & $c_v = \frac{3}{2} k$

$\mapsto$ $\boxed{\gamma = 5/3}$

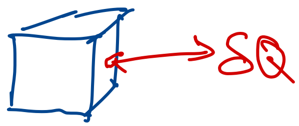
- this is often called adiabatic EOS

$$\boxed{P = K \rho^{5/3}}$$ (no heat exchange)

- relation between pressure and internal energy

$$\boxed{P = (\gamma - 1) e}$$ (invalid for $\gamma = 1$)

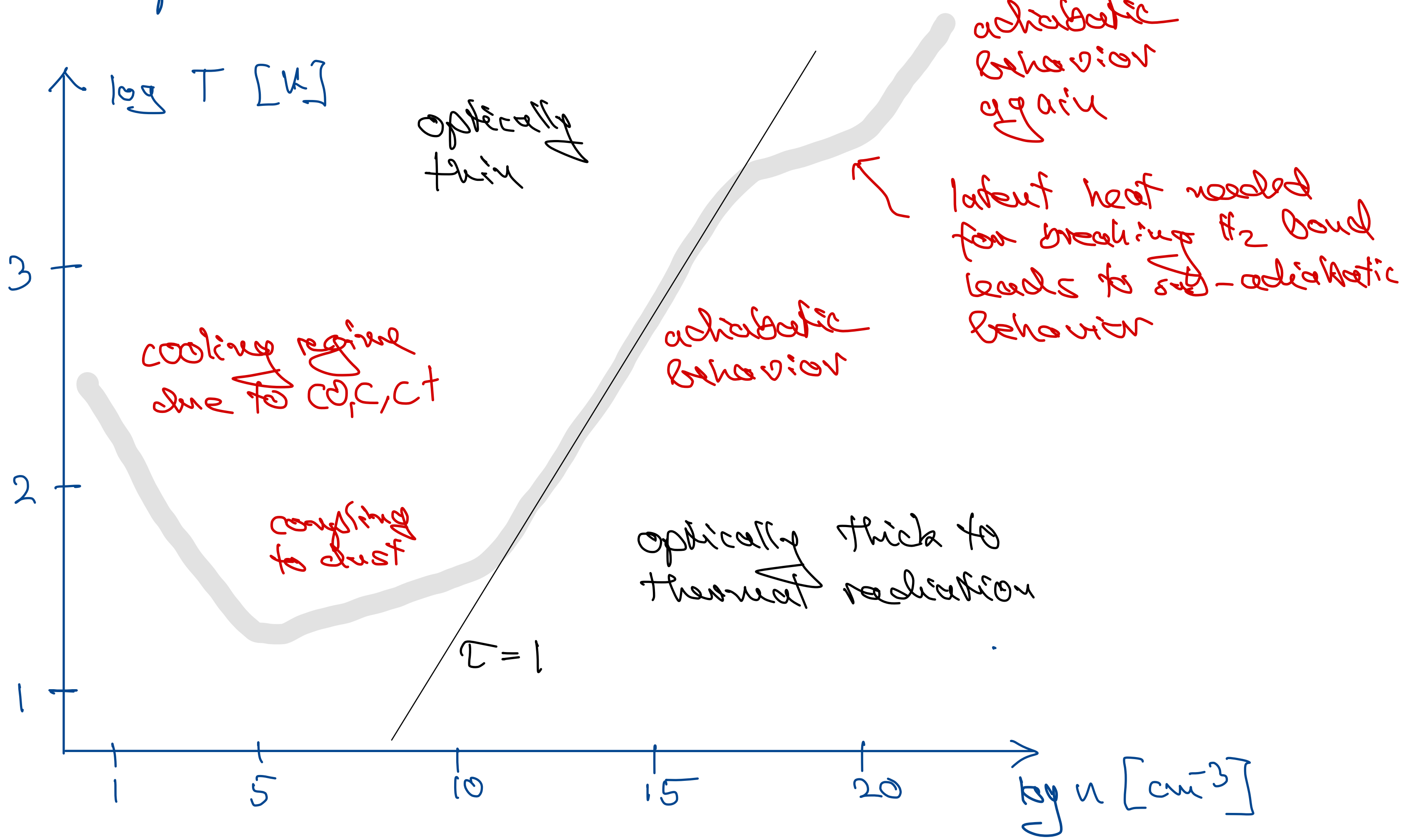
- isothermal EOS: $\gamma = 1$ & $K = c_s^2$ $\mapsto$ $\boxed{P = c_s^2 \rho}$



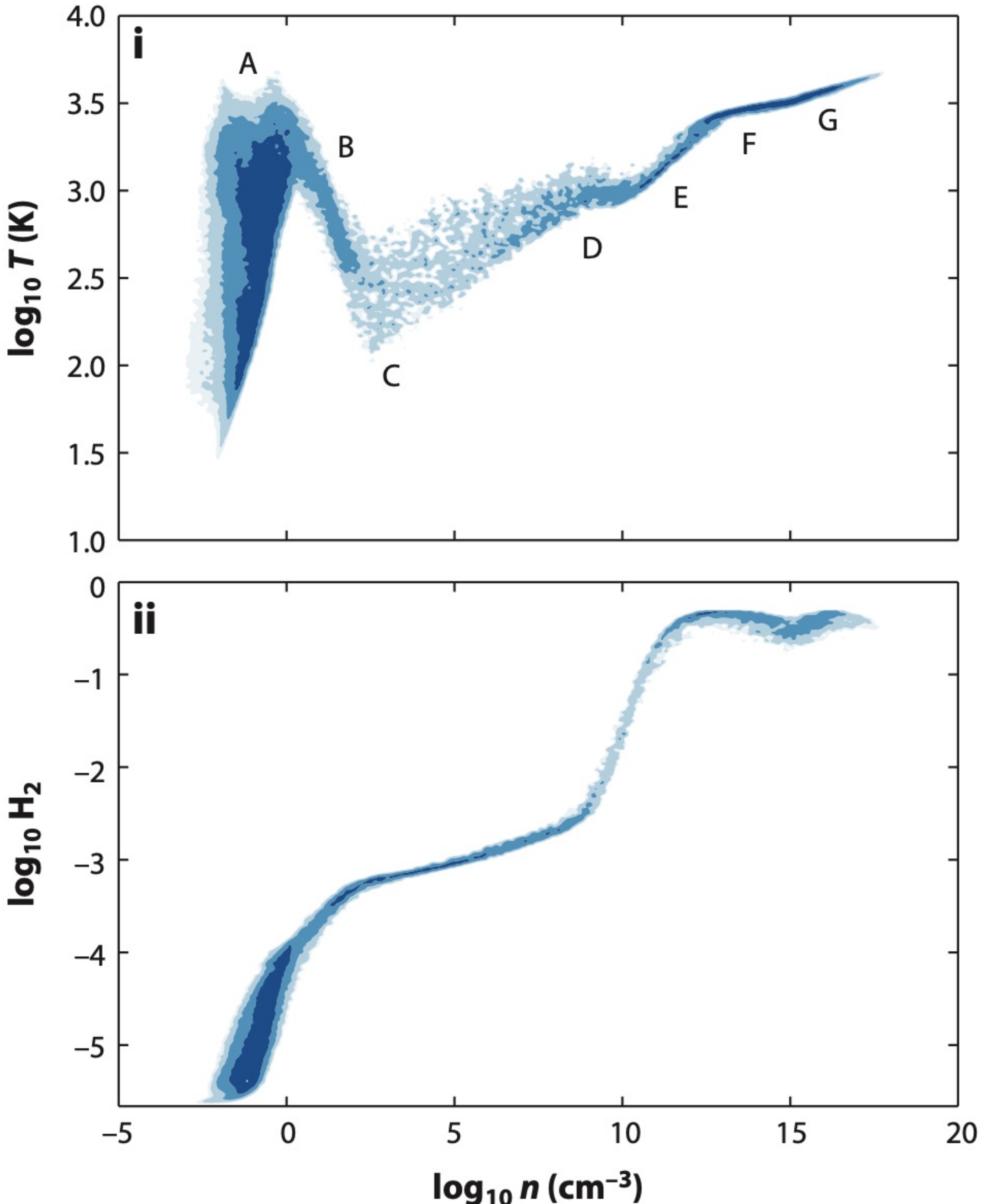
.. classically: achieved by perfect coupling with external heat bath

.. astrophysics: coupling to internal degrees of freedom

1 example: ISM at $z = 20$



\ example: primordial gas ($Z = 0$)



from Klessen & Glover (2023, ARAA, 61, 65)

- implicit relation between T and n:

$$\boxed{\gamma = 1 + \frac{d \log T}{d \log n}}$$

- effective equation of state:
  - $\rightarrow$ solve for $P = K\rho^{\gamma}$ and neglect energy equation

- alternative:

  solve for $P = \frac{k}{\mu m_p} \cdot \rho T$ AND
  energy equation AND
  pick correct μ

* excursion: how to compute mean mol. weight?

- in (partially) ionized gases (of mixed composition) ions and electrons contribute equally to the pressure, but not to the mass

- the "correction" factor is called mean molecular weight μ

- distinguish between hydrogen X, helium Y, and all heavier elements (metals) Z, with X, Y, Z being mass fractions.

$\rightarrow$ $$\boxed{X + Y + Z = 1}$$

- in fully ionized plasma

  - hydrogen contributes <u>1 e^- per nucleus</u> ($_1H$)

  - helium contributes <u>2 e^- per 4 nuclei</u> (^4_2He)

    [note: $^4_2$ Helium has 2 protons ($Z=2$) and 4 nuclei ($A=4$)

  - metals contribute <u>N electrons</u> (1 per proton) and there are typically as many neutrons as protons in the nucleus.

    $\hookrightarrow$ $N \sim \frac{1}{2} A$

  - the mass of a nucleon (p^+ or n^0) is $m_p = 1{,}67 \cdot 10^{-24}$ g

    [note: for the Sun the average mass of all metals is $\sim 16\, m_p \to \bar{A} \sim 16$.]

\ the number density of nuclei in solar-type star is

$$\boxed{n_i \approx \frac{\rho}{m_p}\left[X + \frac{1}{4}Y + \frac{1}{16}Z\right]} \quad (1)$$

similarly, the number density of electrons

$$\boxed{n_e \approx \frac{\rho}{m_p}\left[X + \frac{1}{2}Y + \frac{1}{2}Z\right]} \quad (2)$$

from X: $1e^-$; Y: $\frac{2}{4}e^-$; Z: $\frac{N}{A}e^- \approx \frac{1}{2}e^-$ per nucleon

simplify: $X + \frac{1}{2}Y + \frac{1}{2}Z = \frac{X}{2} + \frac{1}{2}\underbrace{(X+Y+Z)}_{1} = \frac{1}{2}(1+X)$

$$\hookrightarrow \quad n_e = \frac{\rho}{m_p}\left[\frac{1}{2} + \frac{X}{2}\right]$$

- the total number of particles contributing to pressure is then from 1 + 2:

$$\boxed{n = n_i + n_e \approx \frac{\rho}{m_p}\left[2X + \frac{3}{4}Y + \frac{1}{2}Z\right]}$$

[where we have dropped the $\frac{z}{16}$ contribution from heavy nuclei]

- with the definition $\boxed{\rho = \mu m_p n}$ (3) we get

$$\boxed{\mu = \frac{1}{2X + \frac{3}{4}Y + \frac{1}{2}Z}}$$

mean molecular weight

- values

- atomic hydrogen gas (no free e^-, $Y = Z = 0$)

  $\hookrightarrow$ $\mu = 1$

- pure fully ionized hydrogen plasma ($Y = Z = 0$)

  $\hookrightarrow$ $\mu = \frac{1}{2}$

- deep stellar interior ($X = 0{,}71$, $Y = 0{,}27$, $Z = 0{,}02$)

  $\hookrightarrow$ $\mu^{-1} = \left(2 \cdot 0{,}71 + \frac{3}{4} \cdot 0{,}27 + \frac{1}{2} \cdot 0{,}02\right)^{-1}$

  $= 1{,}63^{-1}$

  $\hookrightarrow$ $\mu = 0{,}61$

\ in ISM:

.. cold molecular hydrogen gas (no free e^-, $Y=Z=0$)

$$n = n_i = \frac{\rho}{m_p} \cdot \left[\frac{1}{2} \cdot X \right]$$

and from ③, we get $\mu = 2$

.. molecular hydrogen with helium (no e^-, $X \approx 0.7$, $Y \approx 0.3$)

$$n = n_i = \frac{\rho}{m_p} \underbrace{\left[\frac{1}{2} \cdot X + \frac{1}{4} \cdot Y \right]}_{\mu^{-1}} = \frac{\rho}{m_p} \cdot \underbrace{0.425}_{1/2.35}$$

$\longrightarrow$ $\mu = 2.35$

.. atomic hydrogen with helium (no e^-, $X \approx 0.7$, $Y \approx 0.3$)

$$n = n_i = \frac{\rho}{m_p} \left[X + \frac{1}{4} Y \right] = \frac{\rho}{m_p} \cdot 0.775$$

$\longrightarrow$ $\mu = 1.3$

* competition between gravity and opposing forces:

↳ **VIRIAL BALANCE EQUATION**

- start with Euler equation:

$$\rho \partial_t v_i = \underbrace{-\partial_i P}_{\text{thermal pressure}} - \underbrace{\rho \partial_i \Phi}_{\text{gravity}} - \underbrace{\frac{1}{8\pi} \partial_i B_j B_j}_{\text{magnetic pressure}} + \underbrace{\frac{1}{4\pi} B_j \partial_j B_i}_{\text{magnetic tension}}$$

- multiply with x_i and $\int dV$:

$\boxed{\text{LHS}}$

$$\int \rho x_i \partial_t v_i \, dV = \int \rho x_i \partial_t^2 x_i \, dV$$

$$= \int \rho \partial_t (x_i \partial_t x_i) dV - \int \rho \partial_t x_i \cdot \partial_t x_i \, dV$$

$$= \int \rho \partial_t^2 \left(\frac{x_i x_i}{2}\right) dV - \int \rho v_i v_i \, dV$$

$$= \frac{1}{2} \partial_t^2 \int \rho x_i x_i \, dV - 2 \cdot \int \frac{1}{2} \rho v_i v_i \, dV$$

$$= \frac{1}{2} \ddot{I} - 2T$$

with $I = \int \rho r^2 dV$ = moment of inertia

[note: in principle requires rotation axis]

and $T = \frac{1}{2} \int \rho v^2 dV$ = bulk kinetic energy

[note: random thermal motion excluded]

$\boxed{\text{RHS}}$

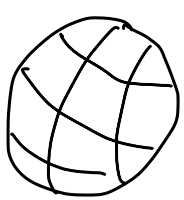
first term: $-\int x_i \partial_i P \, dV = -\underbrace{\int \partial_i (x_i P) \, dV}_{\text{div}(\vec{x} P) \to \text{Gauss}} + \int P \underbrace{\partial_i x_i}_{1+1+1=3} dV$

$$= -\oint P x_i \, dS_i + 3 \int P \, dV$$

$$= -2T_s + 2U$$

with $U = \frac{3}{2} \int P \, dV$ = pressure integral

[note: this is NOT the internal energy!]

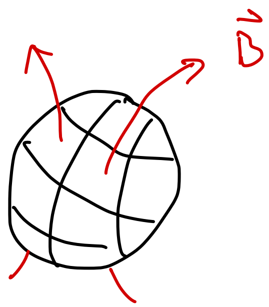


T_s = pressure surface term $= \frac{1}{2} \oint P \vec{r} \cdot d\vec{S}$

<u>second term:</u> $\int \rho x_i \partial_i \Phi \, dV = \int \rho \vec{r} \cdot \vec{F}_g \, dV = W$

with W = total potential energy

<u>third term:</u> $-\frac{1}{8\pi}\int x_i \partial_i B^2 dV = -\frac{1}{8\pi}\int \underbrace{\partial_i (x_i B^2)}_{\mathrm{div}(\vec{x} B^2) \to \text{Gauß}} dV + \frac{1}{8\pi}\int B^2 \underbrace{\partial_i x_i}_{1+1+1=3} dV$



$$= \underbrace{-\frac{1}{8\pi}\oint B^2 \vec{r} \, d\vec{S}}_{\text{vanishes!}} + \frac{3}{8\pi}\int B^2 dV = 3M$$

with magnetic energy $M = \frac{1}{8\pi}\int B^2 dV$

<u>fourth term:</u> $\frac{1}{4\pi}\int x_i B_j \partial_j B_i \, dV = T_M - 2M$

with $T_M = \oint x_i B_i B_j dS_j$ = magnetic stress at surface

[note: $\frac{1}{4\pi}\int x_i B_j \partial_j B_i dV = \frac{1}{4\pi}\left\{ \int \underbrace{\partial_j (x_i B_i B_j)}_{\text{Gauß}} dV - \int \underbrace{(\partial_j x_i)}_{\delta_{ij}} B_i B_j dV - \int \underbrace{(\partial_j B_j)}_{\mathrm{div}\,\vec{B} = 0} x_i B_i dV \right\}$

$$= \frac{1}{4\pi}\oint x_i B_i B_j dS_j - \frac{2}{8\pi}\int B^2 dV = T_M - 2M$$]

- all together: SCALAR VIRIAL THEOREM

$$\boxed{\cancel{\frac{1}{2}\ddot{I}} = 2(\cancel{T} - \cancel{T_s}) + 2U + W + \cancel{M} + \cancel{T_M}}$$

- simplifications
  - neglect magnetic fields
  - neglect surface terms
  - consider the center of mass system
  - steady state
- virial equilibrium of self-gravitating gases

$$\boxed{2U + W = 0}$$

with pressure integral $U = \frac{3}{2}\int P\,dV$

and potential energy $W = \frac{1}{2}\int \rho\,\Phi\,dV$

- note: relation between U and U_{int} = internal energy
  - each degree of freedom contributes $\frac{1}{2}kT$ per particle
  - ideal monoatomic gas has only translational degrees of freedom
    - $\epsilon_{int} = \frac{3}{2} nkT$ $\quad$ n = number density
  - at same time: ideal EOS:

$$P = nkT$$

  - with $U_{int} = \int \epsilon \, dV$ it follows

$$\boxed{U = U_{int}}$$

in this case

- for ideal monoatomic gases ($\gamma = 5/3$) we also have:

$$\gamma \epsilon = \frac{5}{3} \cdot \frac{3}{2} nkT = \frac{5}{2} nkT = \left(\frac{3}{2}+1\right) nkT = \epsilon + P$$

$\hookrightarrow$ $\boxed{P = (\gamma - 1)\epsilon}$ which is another very useful form of EOS

[recall definition $\gamma = \frac{C_P}{C_V}$]

- use definitions to get: $\boxed{u = \frac{3}{2}(\gamma - 1) U_{int}}$ (*)

* total energy of the system is

$$\boxed{E_{tot} = U_{int} + W}$$

- combine (*) with virial equilibrium $(2U + W = 0)$

$\Rightarrow \quad 3(\gamma - 1)\, U_{int} + W = 0$

- insert into total energy equation

$$\Rightarrow \quad E_{tot} = U_{int} - 3(\gamma-1)\, U_{int}$$
$$= -(3\gamma - 4)\, U_{int}$$

- because $U_{int} > 0$, to get bound systems ($E_{tot} < 0$) we require

$$\boxed{\gamma > \frac{4}{3}}$$

- no bound equilibria for EOS with $\gamma < 4/3$!

[compare also to Georges Meynet's lecture]

* what happens if $\gamma \to 4/3$?

\ consider full VT:

$$\frac{1}{2}\ddot{I} = 2U + W = 3(\gamma-1)U_{int} + W =$$

$$= 3(\gamma-1)(E_{tot} - W) + W = 3(\gamma-1)\underline{E_{tot}} - (3\gamma-4)W$$

\ for E_{tot}

$$E_{tot} = \cancel{\frac{\gamma-4/3}{\gamma-1}}\, \underbrace{W}_{<0} + \underbrace{\frac{1}{3(\gamma-1)}\cdot\frac{1}{2}}_{\gamma\to 4/3 \Rightarrow >0}\,\underbrace{\ddot{I}}_{\ddot{I}<0!}$$

$<0!$

\ if we want $E_{tot} < 0$ then $\ddot{I} < 0$, the system needs to contract

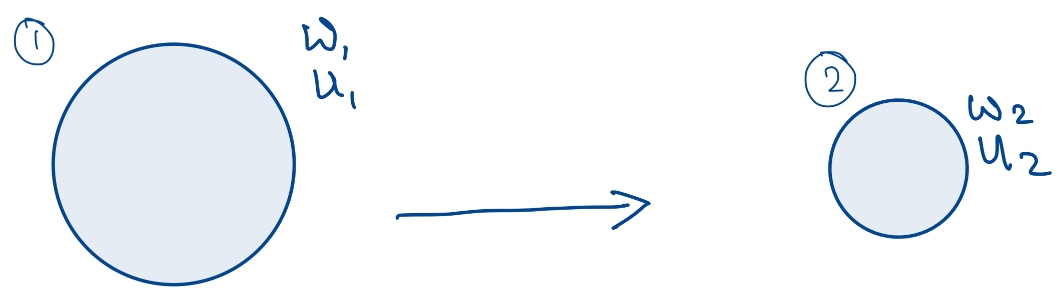
↳ unstable, system goes into collapse.

* strongly self-gravitating systems have <u>negative heat capacity</u> !

↳ this hold for systems close to equilibrium

↳ $\boxed{\text{as system cools, it gets hotter}}$

\ stars try to obey (*) and energy equation (***) at the same time



\ VE: $2U = -W \longrightarrow U = -\frac{W}{2}$

in more compact state $W_2 < W_1$

↳ $U_2 > U_1 \longrightarrow T_2 > T_1$

- interpretation:
  - system shrinks to compensate for energy loss at its surface
  - this releases potential energy
  - in (quasi) equilibrium, half of this energy is used to compensate for energy loss at surface, and other half goes into internal energy, increasing the temperature!
  - this energy is received by outer envelope
  - development of core-halo structure
- note: this applies equally well to star clusters
  → gravothermal catastrophy!
- all strongly self-gravitating systems (close to equilibrium) behave this way

ENJOY the WEEKEND