Les Houches

Physics of Star Formation

February 12-23, 2024

brief theoretical introduction

— basic equations —

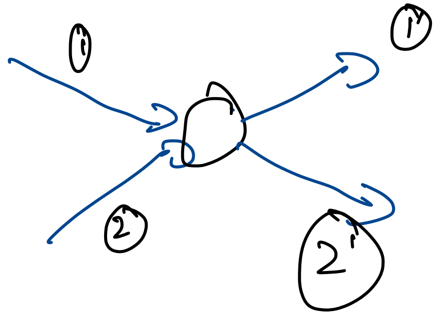
Rolf Klessen

Feb. 13, 2024

MOTIVATION OF BASIC EQUATIONS

* equations of hydrodynamics

* reflect conservation laws; these in turn reflect symmetries in fundamental processes;
fully elastic scattering on particle level

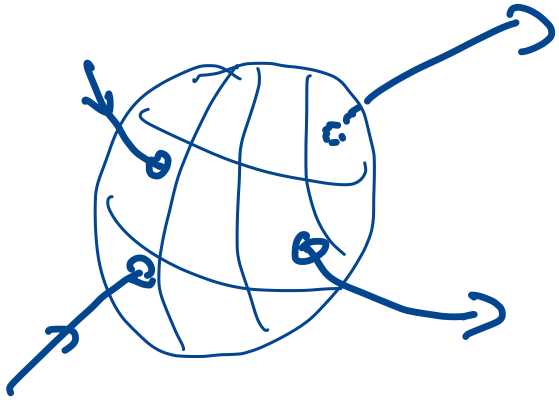


five conserved quantities
1) mass
2) - 4) momentum
5) energy

* formulation of conservation laws

$$\frac{\partial}{\partial t}\left[\text{quantity}\right] + \vec{\nabla}\left[\text{flux of quantity}\right] = 0$$

in the absence of collision terms $= \cancel{\left[\text{collision term}\right]}$



* there are two approaches

[quantity] per unit volume

↳ density [cm^{-3}]

[quantity] per unit mass

↳ specific quantity [g^{-1}]

* <u>conservation of mass:</u>

↳ continuity equation

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho \vec{v}) = 0} \quad (1)$$

sometimes this better formulated in index notation

[Einstein's sum convention

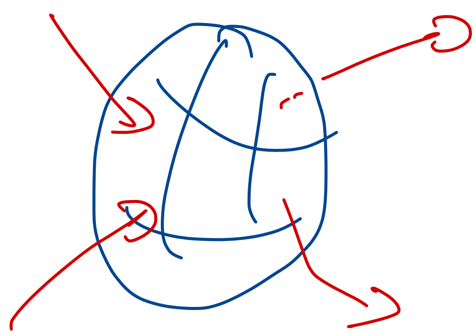
$\partial_i v_i = \sum_{i=1} \partial_i v_i = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$]

$$\partial_t \rho + \partial_i(\rho v_i) = 0$$

$$\partial_t \rho + \underbrace{\partial_j \rho v_j}_{\text{product rule}} = \partial_t \rho + (\partial_j \rho) v_j + \rho (\partial_j v_j) = 0$$

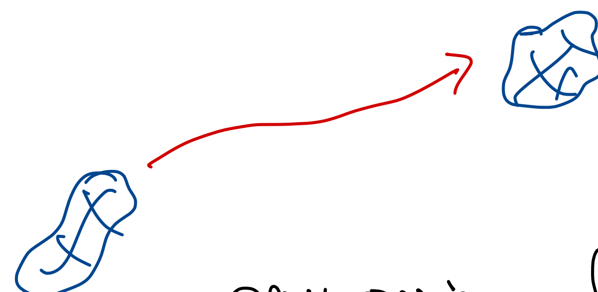
$$\boxed{\frac{\partial \rho}{\partial t} + \vec{v}\cdot\vec{\nabla}\rho = -\rho\, \vec{\nabla}\cdot\vec{v}} \quad (1) \longrightarrow \frac{d\rho}{dt} = -\rho\, \vec{\nabla}\cdot\vec{v}$$

* difference between comoving and fixed coordinates:



fixed frame

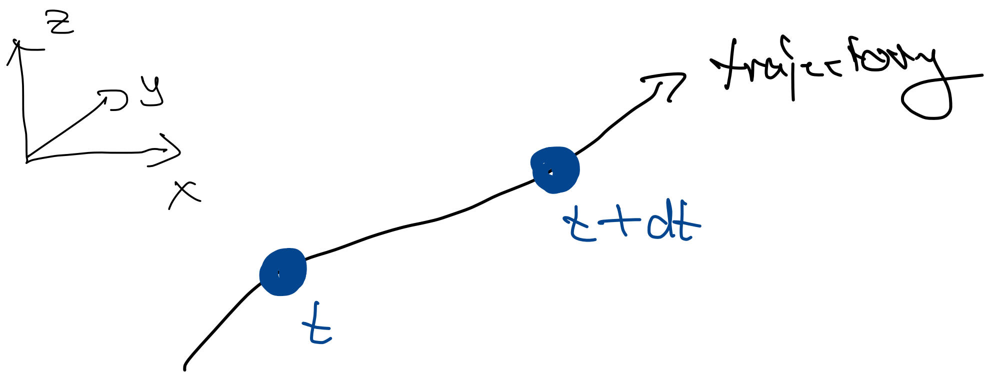
Eulerian point of view



comoving frame

Lagrangian view point

NB: recall how to compute changes along trajectory



example of number density $n(\vec{x}, t)$

$$dn(\vec{x},t) = \frac{\partial n}{\partial x_i}\, dx_i + \frac{\partial n}{\partial t}\, dt \quad \Big| \; \frac{d}{dt}$$

$$\frac{dn(\vec{x},t)}{dt} = \underbrace{\frac{\partial n}{\partial x_i}}_{\vec{\nabla} n \,\cdot}\; \underbrace{\frac{dx_i}{dt}}_{\vec{v}} + \frac{\partial n}{\partial t}\, \underbrace{\cancel{\frac{dt}{dt}}}_{1} \Longrightarrow \boxed{\frac{dn}{dt} = \frac{\partial n}{\partial t} + \vec{v}\cdot\vec{\nabla} n}$$

① $$\boxed{\frac{\partial \rho}{\partial t} + \vec{v}\cdot\vec{\nabla}\rho = \frac{d\rho}{dt} = -\rho\, \vec{\nabla}\cdot\vec{v}}$$

continuity equation

* conservation of momentum: Navier-Stokes equation
Euler equation

$$\boxed{\partial_t \rho v_i + \partial_j T_{ij} = 0}$$

momentum density

stress-energy tensor

\ gives acceleration of fluid element

\ on "RHS" we "collect" forces that act on fluid element $\longrightarrow$ Newton 3 for fluids!

\ let us reformulate to separate acceleration

Lagrangian view point:

$$\frac{d\vec{v}}{dt} = \text{sum of all forces}$$

$$= \text{pressure} + \text{viscose} + \text{grav.} + \text{magn. forces} + \ldots$$

- $$\underbrace{\frac{d\vec{v}}{dt}}_{\text{acceleration of fluid element}} = \frac{\partial \vec{v}}{\partial t} + \underbrace{\vec{v}\cdot\vec{\nabla}\,\vec{v}}_{\text{advection term; quadratic in velocity, non-linear term}} = \sum \text{forces}$$

advection term; quadratic in velocity, non-linear term $\mapsto$ gives rise to fluid instabilities

$$= \underbrace{-\frac{1}{\rho}\vec{\nabla}P}_{\text{force due to pressure gradients}}$$

$$+ \underbrace{\frac{1}{\rho}\left[\eta\,\vec{\nabla}^2\vec{v} + \left(\zeta + \frac{\eta}{3}\right)\vec{\nabla}\left(\vec{\nabla}\cdot\vec{v}\right)\right]}$$

viscous forces: $\vec{\nabla}^2 v \mathrel{\hat{=}}$ compression of flow lines; $\vec{\nabla}(\vec{\nabla}\cdot\vec{v}) \mathrel{\hat{=}}$ bending of flow lines

- neglecting viscosity: Euler's equation
- viscosity leads to dissipation!

NB: $\vec{\nabla}^2 \vec{v} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \longrightarrow$ "compression"

$\vec{\nabla}(\vec{\nabla}\cdot\vec{v}) = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) \longrightarrow$ "bending"

NB: these equations are obtained as the first three central velocity moments of Boltzmann equation:

$$f = \text{phase space density} = f(\vec{q}, \vec{p}, t) = f(\vec{x}, \vec{v}, t)$$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \vec{v}\cdot\vec{\nabla}_x f + \underbrace{\vec{a}}_{\text{acceleration: } \vec{F}}\cdot\vec{\nabla}_v f = \cancel{\dot{f}_c} = 0$$

in collision-less Boltz-mann equ.

* when to neglect viscosity?

$$\frac{1}{\rho}\Big[\underbrace{\eta\, \vec{\nabla}^2 \vec{v}}_{\nu\, \vec{\nabla}^2 \vec{v}} + \cancel{\left(\xi + \frac{\eta}{3}\right)\vec{\nabla}(\vec{\nabla}\cdot\vec{v})}\Big]$$

often small $\longrightarrow$ neglect

with kinetic viscosity $\nu = \dfrac{\eta}{\rho}$

$[\nu] = \dfrac{cm^2}{s}$; $[\eta] = \dfrac{g}{cm \cdot s}$

↳ we get for NS equation:

$$\boxed{\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + \underbrace{\vec{v}\cdot\vec{\nabla}\vec{v}}_{\text{advection term: } \frac{v^2}{\ell}} = + \underbrace{\nu \vec{\nabla}^2 \vec{v}}_{\text{viscous term: } \frac{\nu v}{\ell^2}} - \frac{1}{\rho}\vec{\nabla}P}$$

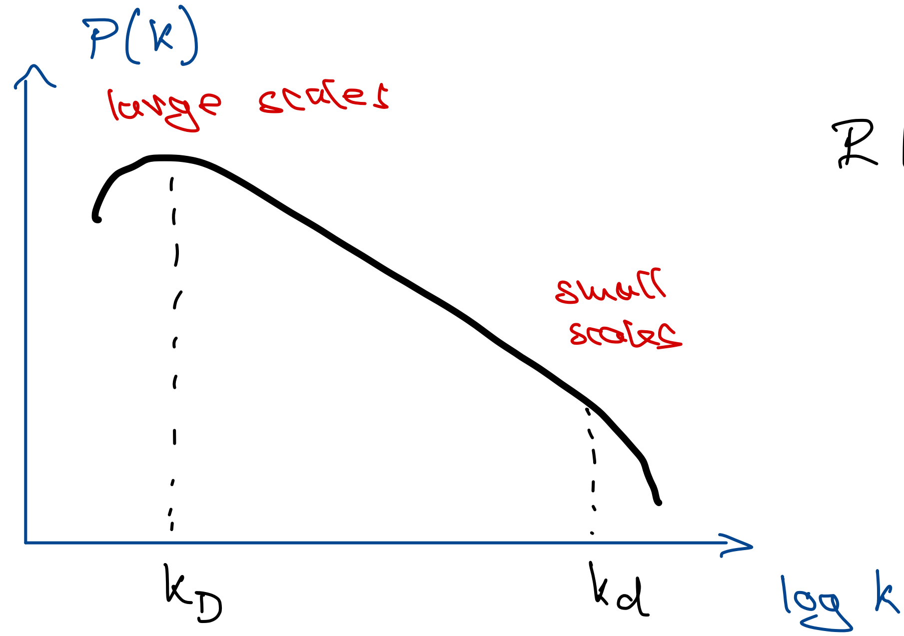
approximate:

$\nabla \sim \dfrac{1}{\ell}$

↳ definition of Reynolds number

$$Re = \frac{\text{advection}}{\text{dissipation}} = \frac{v^2/\ell}{\nu v/\ell^2} = \frac{v \cdot \ell}{\nu}$$

↳ large Re lead to <u>turbulent</u> instability



$P(k)$ = power spectrum of flow (of its kinetic energy)

k = wave number

driving scale

dissipation scale
$k_d \sim 1/\lambda$ with λ = mean-free path

if $Re \gg 100$ flow is turbulent ; advection dominates

if $Re \ll 100$ flow is laminar

if $Re \lesssim 1$ dissipation regime: $E_{kin} \longrightarrow$ thermal energy

$(\ell \sim \lambda)$

* for large Re, or scales $\gg$ mean free path

NS $\longrightarrow$ Euler equation:

$$\boxed{\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\vec{\nabla}\vec{v} = -\frac{1}{\rho}\vec{\nabla}P - \vec{\nabla}\Phi}$$

* adding (self)-gravity:

\ grav. force $\quad F_g = -\vec{\nabla}\Phi$

\ introduce additional equation:

$$\boxed{\vec{\nabla}^2\Phi = 4\pi G\rho}$$

(for self-gravity; otherwise simply add external force)

\ grav. constant: $\quad G = 6{,}67\cdot 10^{-8}\ \frac{cm^3}{g\,s^2}$

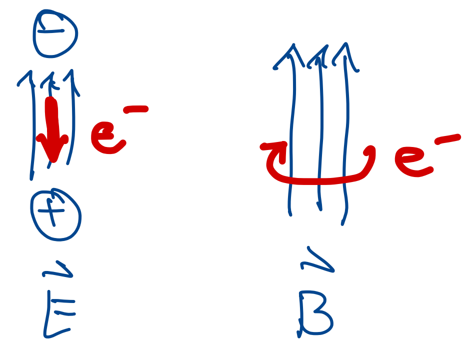
* including magnetic field:

- Maxwell equations:

$$\boxed{\begin{aligned} &\vec{\nabla}\cdot E = 4\pi\rho & \qquad &\vec{\nabla}\cdot\vec{B} = 0 \\ &\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} & \qquad &\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t} \end{aligned}}$$

- force: Lorentz force

$$\vec{F}_L = q\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)$$



- energy density:

$$u = \frac{1}{8\pi}\left(\vec{E}^2 + \vec{B}^2\right)$$

- energy flux: Poynting vector

$$\vec{S} = \frac{c}{4\pi}\left(\vec{E}\times\vec{B}\right)$$

- note units in cgs:

.. $[u] = \frac{\text{erg}}{\text{cm}^2} \longrightarrow [E] = [\vec{B}] = \left(\frac{\text{erg}}{\text{cm}^3}\right)^{1/2} = \frac{g^{1/2}}{\text{cm}^{1/2}\, s}$

recall: $1 \text{ erg} = 1 \frac{g\,\text{cm}^2}{s^2}$

.. force: $[f] = [q] \cdot [E]$

with $[f] = \frac{\text{erg}}{\text{cm}} = \frac{g\,\text{cm}}{s^2}$

$\stackrel{!}{=}$ dyn

$$\left.\right\} \; [q] = \frac{[f]}{[E]} = \frac{\text{erg}/\text{cm}}{(\text{erg}/\text{cm}^3)^{1/2}} = (\text{erg} \cdot \text{cm})^{1/2} = \frac{g^{1/2}\,\text{cm}^{3/2}}{s}$$

$\stackrel{!}{=}$ esu (electrostatic unit)

.. elementary charge:

$e = 4{,}8 \cdot 10^{-10}$ esu

.. Poynting vector:

$$[S] = \left[\frac{c}{4\pi} \vec{E} \times \vec{B}\right] = \frac{\text{cm}}{s} \cdot \frac{\text{erg}}{\text{cm}^3} = \frac{\text{erg}}{s} = \frac{g}{s^3}$$

- ideal MHD approximation
  - strong coupling between neutrals, ions, electrons
  - good conductivity $\longmapsto$ effective charge neutrality ($\epsilon = 0$)
  - frequent collisions ($\lambda_{mfp} \ll L$) $\longmapsto$ coupling between n & e+i

  ↳ <u>single fluid approximation</u>

- note: non-ideal MHD important for star & planet formation

- Ohm's law: $\vec{j} = \sigma \left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B} \right)$ with σ = conductivity

- <u>induction equation:</u> time-evolution of (pre-existing) magnetic field

$$\frac{\partial \vec{B}}{\partial t} = \underbrace{- \frac{c^2}{4\pi\sigma} \vec{\nabla}^2 \vec{B}}_{\text{diffusion term}} + \underbrace{\vec{\nabla} \times (\vec{v} \times \vec{B})}_{\text{advection term}}$$

in ideal MHD limit $\sigma \longmapsto \infty$

↳ $$\boxed{\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times \left(\vec{v} \times \vec{B} \right)}$$

\ back to force equation:

.. force density $\vec{f}_L = \cancel{nq\vec{E}} + \frac{\vec{j}}{c}\times\vec{B}$ with $\vec{j} = nq\vec{v}$

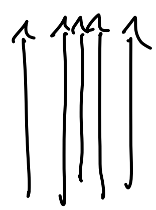
ideal MHD

.. Maxwell: $\vec{j} = \frac{c}{4\pi}\vec{\nabla}\times\vec{B}$

together → $\vec{f}_L = \frac{1}{4\pi}(\vec{\nabla}\times\vec{B})\times\vec{B}$

$$= \frac{1}{4\pi}\underbrace{(B\cdot\vec{\nabla})\vec{B}}_{\text{magnetic tension}} + \underbrace{\frac{1}{8\pi}\vec{\nabla}B^2}_{\text{magnetic pressure}}$$

tension



magn. pressure

magnetic pressure:

$P_{mag} = \frac{\vec{B}^2}{8\pi}$ = energy density in B-field

* together: Euler equation in ideal MHD

$$\boxed{\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\vec{\nabla}\vec{v} = -\frac{1}{\rho}\vec{\nabla}\left(P + \frac{\vec{B}^2}{8\pi}\right) + \frac{1}{4\pi\rho}\left(\vec{B}\cdot\vec{\nabla}\right)B - \vec{\nabla}\Phi}$$

* conservation of energy: [2^{nd} velocity moment of Boltzmann equ.]

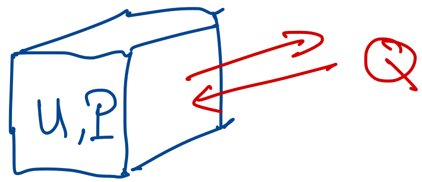
\ in principle, we need to look at energy tensor: $e_{ij} = \frac{1}{2}\rho v_i v_j$

↳ we are mostly interested in the time evolution of internal energy

↳ look at diagonal elements of e_{ij}

↳ $\boxed{e = \frac{1}{2}\rho v^2}$ internal energy density

\ look at 1^{st} law of thermodynamics



$$\boxed{dQ = dU + dW}$$

heat exchange — change of int. energy — associated work

$$dQ = T\,dS \qquad S = \text{entropy}$$

$$dW = P\,dV \qquad T = \text{temperature}$$

P = pressure

V = volume

\ consider specific quantities [per gramm]

$$[\epsilon] = \frac{\text{erg}}{\text{g}} \qquad \text{specific internal energy}$$

$$[s] = \frac{\text{erg}/\text{K}}{\text{g}} \qquad \text{"} - \text{ entropy}$$

[NB: cgs $\longrightarrow$ erg: $\frac{g\,cm^2}{s^2} = 10^{-7}\,J$]

$$\longrightarrow \quad \frac{d\epsilon}{dt} = T\frac{ds}{dt} - \frac{dw}{dt}$$

$$= T\frac{ds}{dt} - \frac{P}{m}\frac{dV}{dt}$$

$$= T\frac{ds}{dt} + \frac{P}{\rho^2}\frac{d\rho}{dt}$$

$$= T\frac{ds}{dt} - \frac{P}{\rho}\vec{\nabla}\cdot\vec{v}$$

$w = \frac{W}{m}$

$s = \frac{S}{m}$

$V = \frac{m}{\rho}$

$dV = -\frac{m}{\rho^2}d\rho$

continuity:

$\frac{d\rho}{dt} = -\rho\,\vec{\nabla}\cdot\vec{v}$

* energy equation:

$$\boxed{\frac{d\epsilon}{dt} = \frac{\partial\epsilon}{\partial t} + \vec{v}\cdot\vec{\nabla}\epsilon = \underbrace{T\frac{ds}{dt}}_{①\,②} + \frac{P}{\rho^2}\frac{d\rho}{dt}}$$

① source terms (heating / cooling) are incorporated here

② isentropic behavior ($dQ = 0$) $\longrightarrow$ zero

* this needs a closure!
(more variables in system than equations)

↳ equation of state (EOS) $\boxed{P = P(\rho, T)}$

- for ideal gas:

$$\boxed{PV = NkT} \quad \hat{=} \quad \boxed{P = \frac{\rho k T}{\mu \cdot m_p}}$$

k = Boltzmann constant
$= 1{,}38 \cdot 10^{-16}$ erg/K

N = number of particles

V = volume

T = temperature

P = pressure

μ = mean molecular weight

m_p = proton mass
$= 1{,}67 \cdot 10^{-24}$ g

* other ways to describe this relation:

- barotropic EOS $\boxed{P = P(\rho)}$

- note: both EOS descriptions can be valid; this implies an implicit relation between ρ and T

- polytropic EOS $\boxed{P = K\rho^{\gamma}}$

γ = polytropic exponent = $\dfrac{\text{specific heat at } P = \text{const.}}{\text{specific heat at } V = \text{const.}}$

$= \dfrac{c_p}{c_v}$

recall: for ideal monoatomic gases

$c_p = c_v + k$ (per particle)

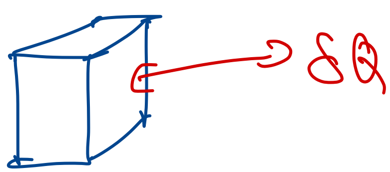
and we have $c_V = \frac{3}{2} k$ and $c_p = \frac{5}{2} k$

↳ we obtain $\boxed{\gamma = \frac{5}{3}}$

- often this is called adiabatic EOS $\boxed{P = K \rho^{5/3}}$

(realized in systems without heat exchange)

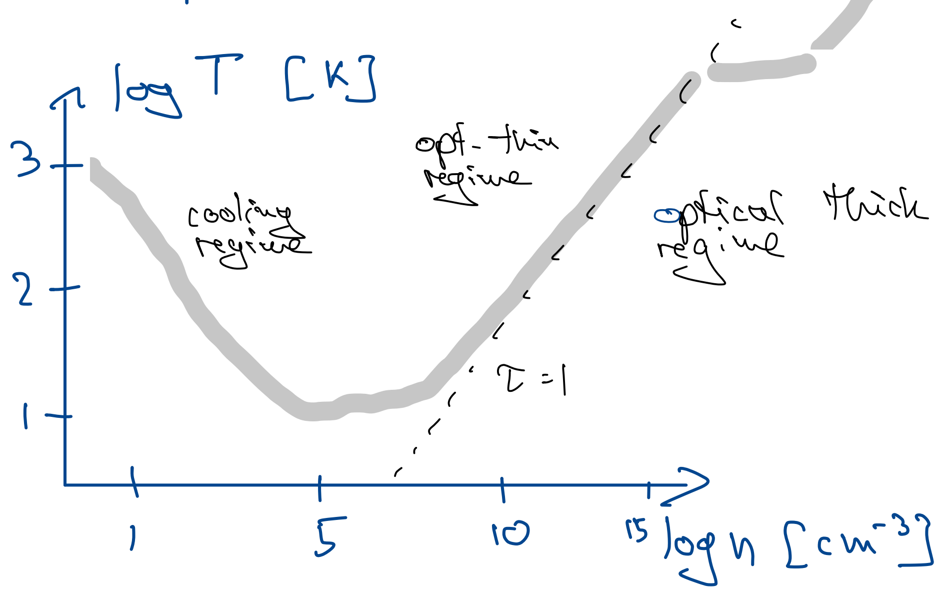
- isothermal EOS: $\gamma = 1$; $K = c_s^2$ $\boxed{P = c_s^2 \rho}$



this can be achieved classically by a perfect coupling of system to external heat bath

in astrophysics: realized by coupling to internal degrees of freedom

- example: ISM (interstellar medium)



CO molecule: highly polar → strong dipole radiation

implicit relation between T and n:

$$\boxed{\gamma = 1 + \frac{d \log T}{d \log n}}$$

effective equation of state!

* summary:

- we have motivated the equations of magnetohydrodynamics for self-gravitating gas, relevant for star & planet formation
- but there is much more!
  - radiation: production, coupling to matter, transport
  - chemical & nuclear reactions
  - energy production & heating / cooling
  - cosmic rays
  - etc...
- most of this will be covered in this school

→ **enjoy your time here!**