The physics of star formation
12-23 Feb 2024 Les Houches (France)

Fundamentals of MHD Turbulence (Part 2)

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Les Houches, February 2024

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MHD spectra

Kolmogorov spectrum

'Strong' or 'eddy' HD turbulence

Two-point correlation

$$R_{ij}(\boldsymbol{\ell}) \equiv \langle u_i(\mathbf{x}) u_j(\mathbf{x}+\boldsymbol{\ell})\rangle$$

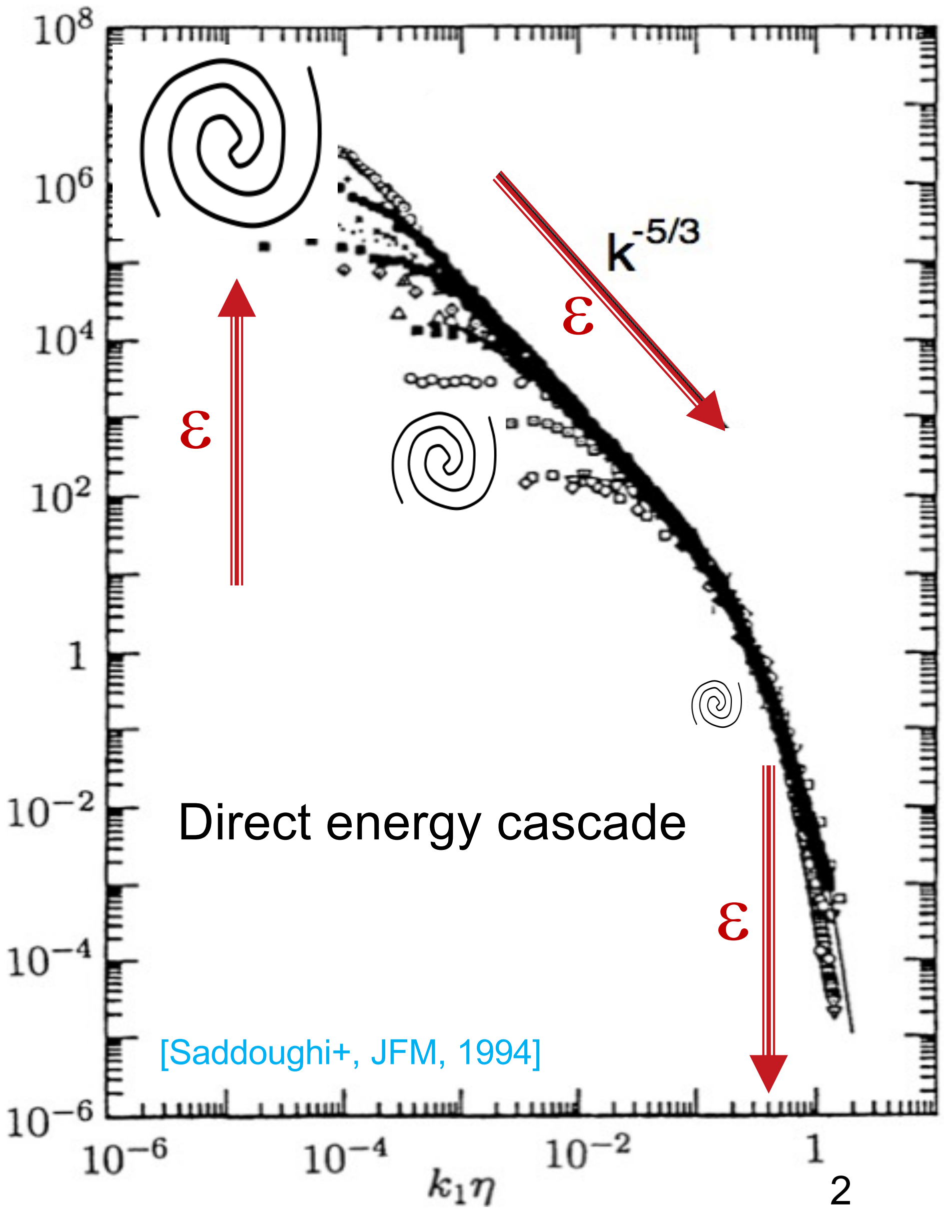
homogeneous turbulence

$$\Phi_{ij}(\mathbf{k}) \equiv \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} R_{ij}(\boldsymbol{\ell})\, e^{-i\mathbf{k}\cdot\boldsymbol{\ell}} d\boldsymbol{\ell}$$

$$E(\mathbf{k}) \equiv \frac{1}{2}\Phi_{ii}(\mathbf{k})$$

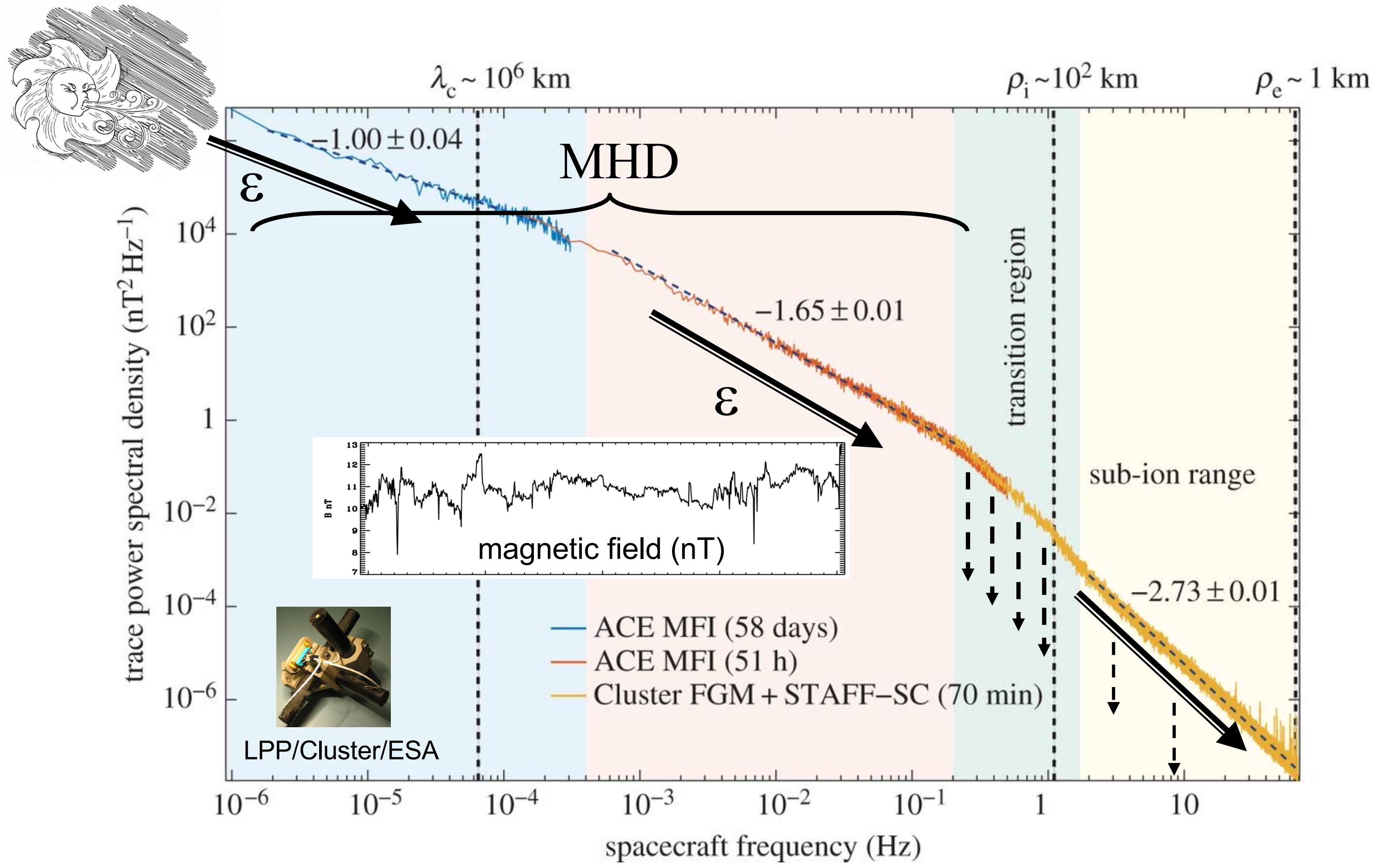
1D energy spectrum

$$E(k) = \oiint E(\mathbf{k})\, dS(\mathbf{k})$$



Solar wind turbulence

At 1 AU

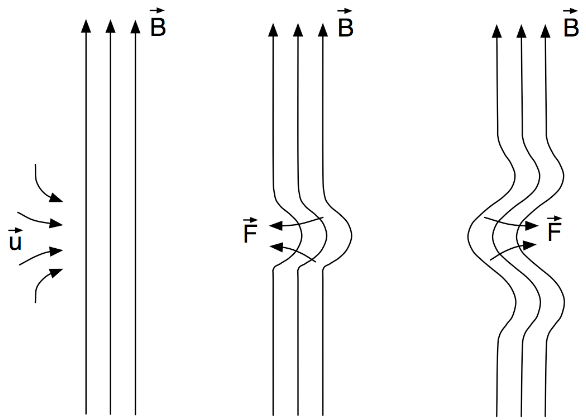


[Kiyani+, Phil. Trans. R. Soc. A, 2015]

Collisions of Alfvén waves

$\mathbf{z}^{\pm} \equiv \mathbf{u} \pm \mathbf{b}$ Elsässer fields

Alfvén waves



$$\frac{\partial \mathbf{z}^{\pm}}{\partial t} \mp \mathbf{b}_0 \cdot \nabla \mathbf{z}^{\pm} + \underbrace{\mathbf{z}^{\mp} \cdot \nabla \mathbf{z}^{\pm}}_{\text{collision}} = -\nabla P_* \qquad \text{(no dissipation)}$$

Linear solution: $\boxed{\omega_k^2 = (\mathbf{k} \cdot \mathbf{b_0})^2}$ $\mathbf{b} = \mathbf{B}/\sqrt{\mu_0 \rho_0}$

Incompressible MHD turbulence is the result of collisions between counterpropagating Alfvén waves

Iroshnikov-Kraichnan spectrum

Phenomenology of Alfvén wave turbulence

$z^+ \sim z^- \sim z,$ balanced turbulence

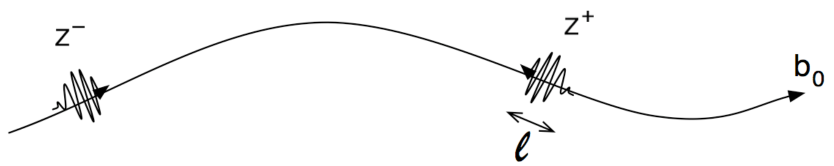
$\tau_A \sim \ell/b_0 \qquad \tau_{NL} \sim \ell/z_\ell \qquad \varepsilon \sim \frac{z_\ell^2}{\omega\tau_{NL}^2} \sim \frac{z_\ell^4}{\ell b_0} \sim \frac{E^2(k)k^3}{b_0},$ (3-wave interactions)

hence the one-dimensional isotropic Iroshnikov–Kraichnan (IK) spectrum of wave turbulence:

$$\boxed{E(k) = C_{IK}\sqrt{\varepsilon b_0}\, k^{-3/2}},$$

[Iroshnikov, SA, 1964; Kraichnan, PoF, 1965]

However, in presence of a uniform magnetic field $\mathbf{b_0}$ incompressible MHD turbulence is anisotropic



Resonance condition

$$\frac{\partial \mathbf{z^s}}{\partial t} - s\mathbf{b_0}\cdot\nabla\mathbf{z^s} = -\mathbf{z^{-s}}\cdot\nabla\mathbf{z^s} - \nabla P_*\,,$$

$$\nabla\cdot\mathbf{z^s} = 0\,. \qquad s = \pm$$

$$z_j^s(\mathbf{x},t) \equiv \int_{\mathbb{R}^3} A_j^s(\mathbf{k},t)\, e^{i\mathbf{k}\cdot\mathbf{x}}\, d\mathbf{k}\,,$$

$$A_j^s(\mathbf{k},t) \equiv \epsilon a_j^s(\mathbf{k},t) e^{-is\omega_k t}\,,$$

$$\frac{\partial a_j^s(\mathbf{k})}{\partial t} = -i\epsilon k_m P_{jn} \int_{\mathbb{R}^6} a_m^{-s}(\mathbf{q})\, a_n^s(\mathbf{p})\, e^{is(\omega_k-\omega_p+\omega_q)t} \underbrace{\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})}_{\text{Triadic interactions}}\, d\mathbf{p}d\mathbf{q}$$

where $P_{jn}(k) \equiv \delta_{jn} - k_j k_n/k^2$

Resonance condition: $\omega_k \equiv k_{||} b_0$

$$\begin{cases} \omega_k &= \omega_p - \omega_q\,, \\ \mathbf{k} &= \mathbf{p}+\mathbf{q}\,, \end{cases} \Longrightarrow \quad q_{||} = 0 \quad \text{and} \quad \mathrm{k}_{||} = \mathrm{p}_{||}$$

Alfvén wave turbulence

this means no cascade along $\mathbf{b_0}$

[Montgomery & Turner, PoF, 1981; Shebalin+, JPP, 1983]

Gives an explanation to observations of anisotropy (in B-confinement)

[Robinson & Rusbridge, PoF, 1971; Zweben+, PRL, 1979]

Iroshnikov-Kraichnan spectrum revisited

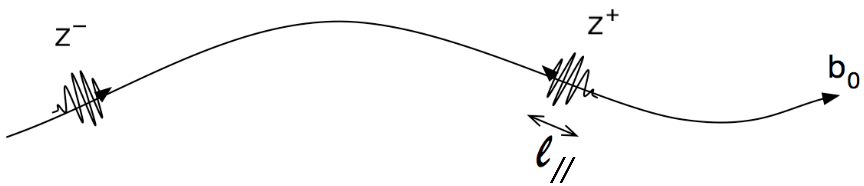
Phenomenology

$\tau_{NL} \sim \ell_\perp / z_\ell$

$\tau_A \sim \ell_\| / b_0$

$z^+ \sim z^- \sim z,$ balanced turbulence

$$\tau_{tr} \sim \omega \tau_{NL}^2 \sim \frac{(\ell_\perp / z_\ell)^2}{\ell_\| / b_0} \sim \frac{k_\| b_0}{k_\perp^2 z_\ell^2} .$$



We deduce from this:

$$\varepsilon \sim \frac{z_\ell^2}{\tau_{tr}} \sim \frac{k_\perp^2 z_\ell^4}{k_\| b_0} \sim \frac{k_\perp^2 (E(k_\perp, k_\|) k_\perp k_\|)^2}{k_\| b_0} \sim \frac{k_\perp^4 k_\| E^2(k_\perp, k_\|)}{b_0} ,$$

hence the two-dimensional anisotropic (axisymmetric) spectrum:

[Ng & Bhattacharjee, PoP, 1997]

$$E(k_\perp, k_\|) \sim \sqrt{\varepsilon b_0}\, k_\perp^{-2} k_\|^{-1/2} .$$

Can be treated analytically (Alfvén wave turbulence) !

Goldreich-Sridhar spectrum

Phenomenology

Conjecture: scale-by-scale balance between τ_A and τ_{NL}

$$\tau_A \sim \tau_{NL}$$

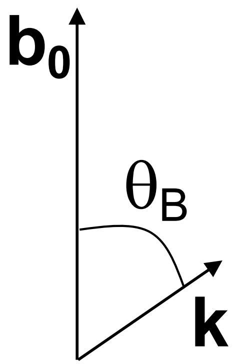
[Higdon, ApJ, 1984; Goldreich & Sridhar, ApJ, 1995]

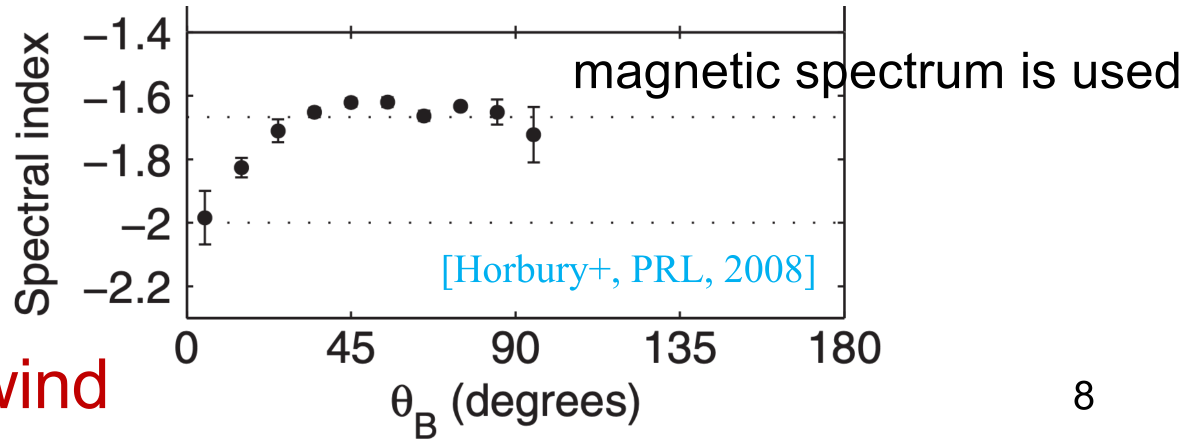
$$\Longrightarrow \quad E(k_\perp) \sim k_\perp^{-5/3}$$

$$E(k_\parallel) \sim k_\parallel^{-2}$$

Anisotropic cascade

$$k_\parallel \sim k_\perp^{2/3}$$

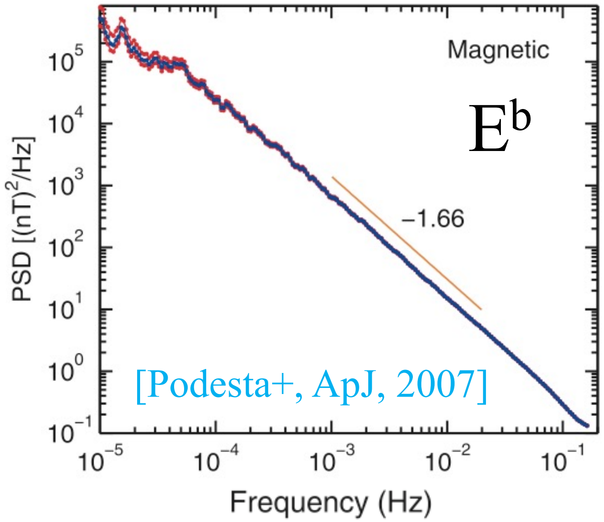


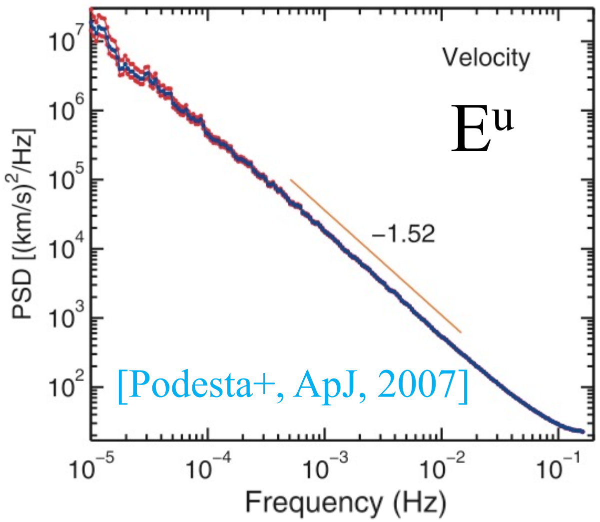


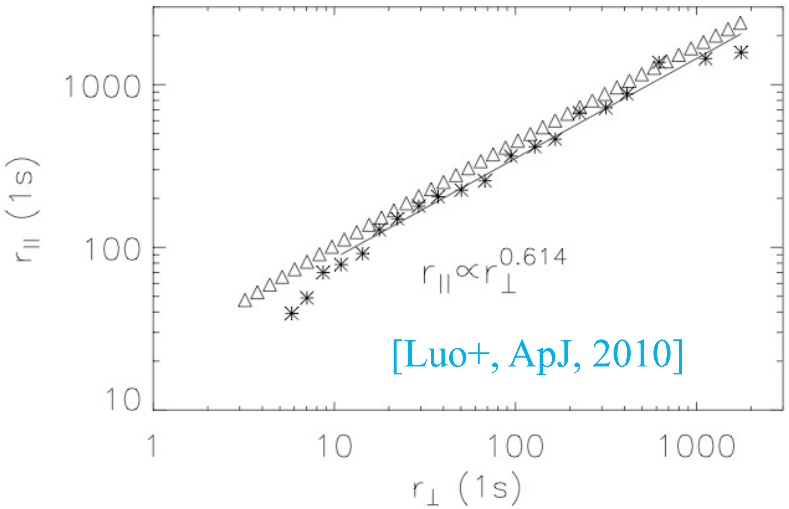
Solar wind

Comparison with data

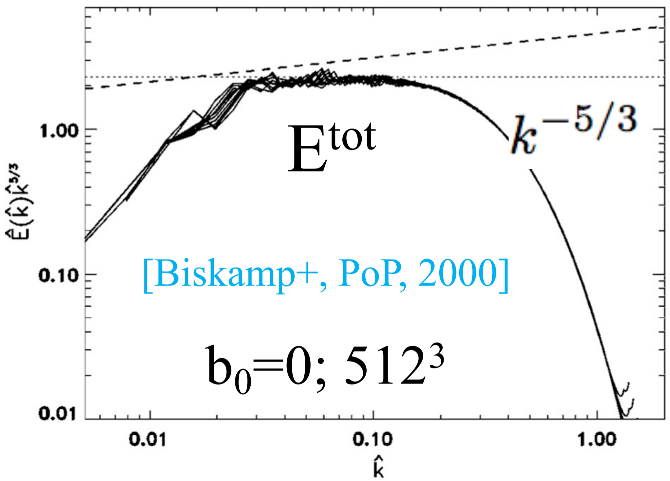
Solar wind data

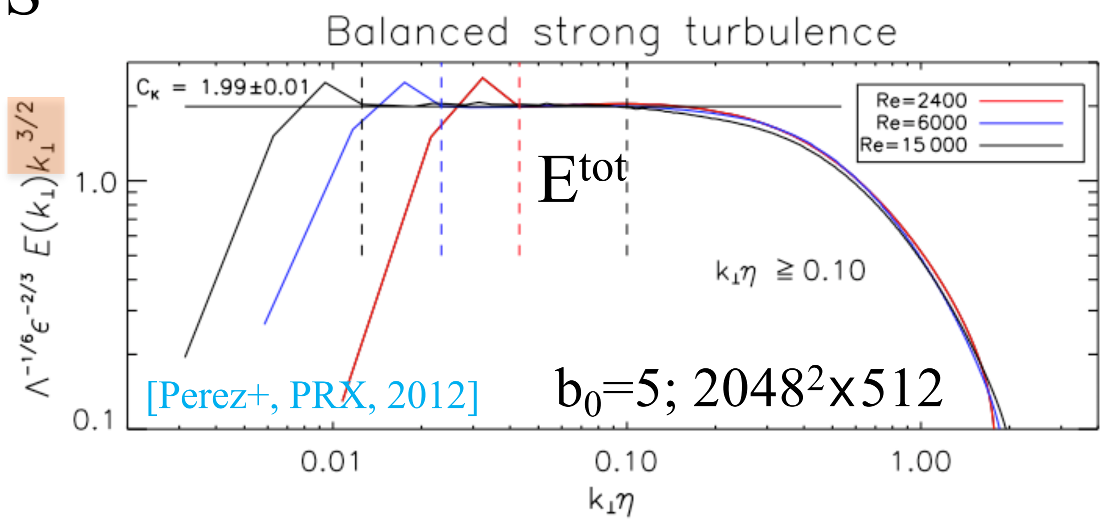






DNS



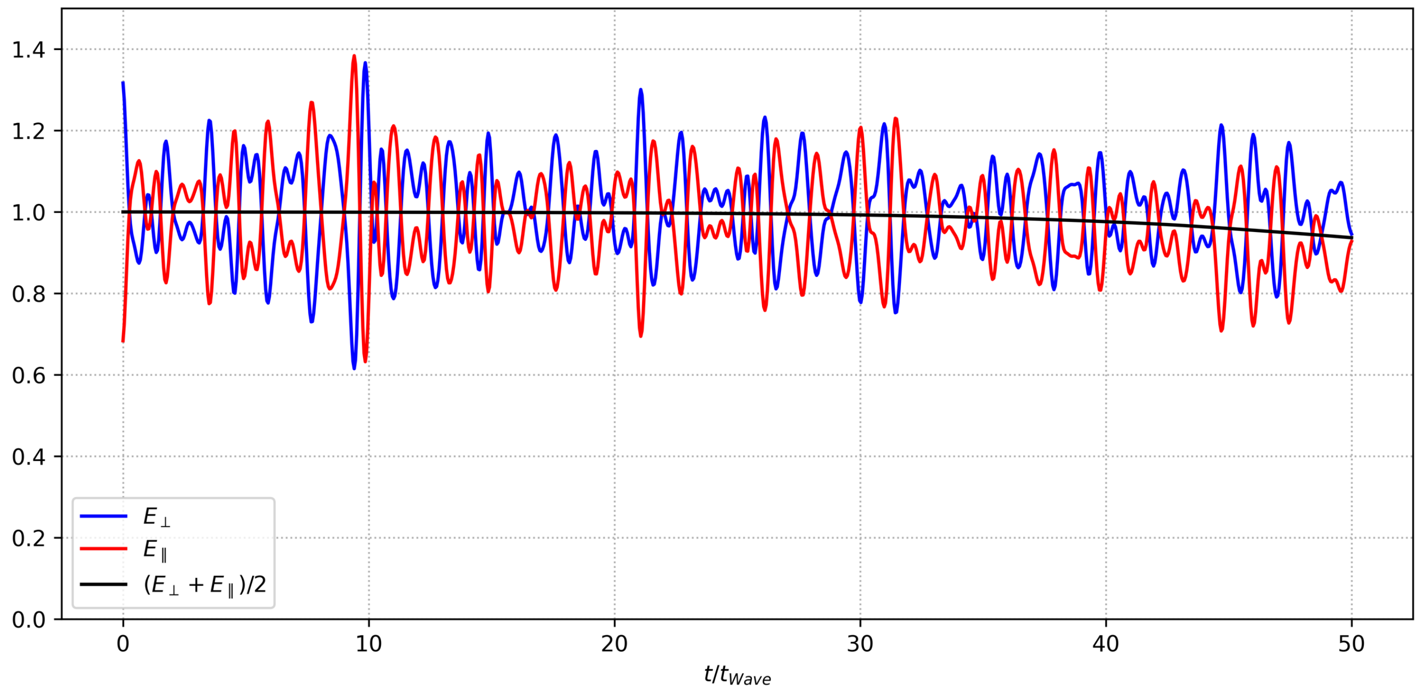


(Alfvén) wave turbulence

What is wave turbulence?

Wave turbulence is the study of the long time statistical behaviour of equations describing a set of weakly nonlinear interacting waves

It is a multiple time scale problem



What is wave turbulence?

Usually, the system is non-isolated having both sources and sinks (of energy or other conserved quantities)

We want to understand eg. the transport properties and how energy might propagate through the **k**-space

But in turbulence, we are faced with the closure problem

Wave turbulence is a solved problem

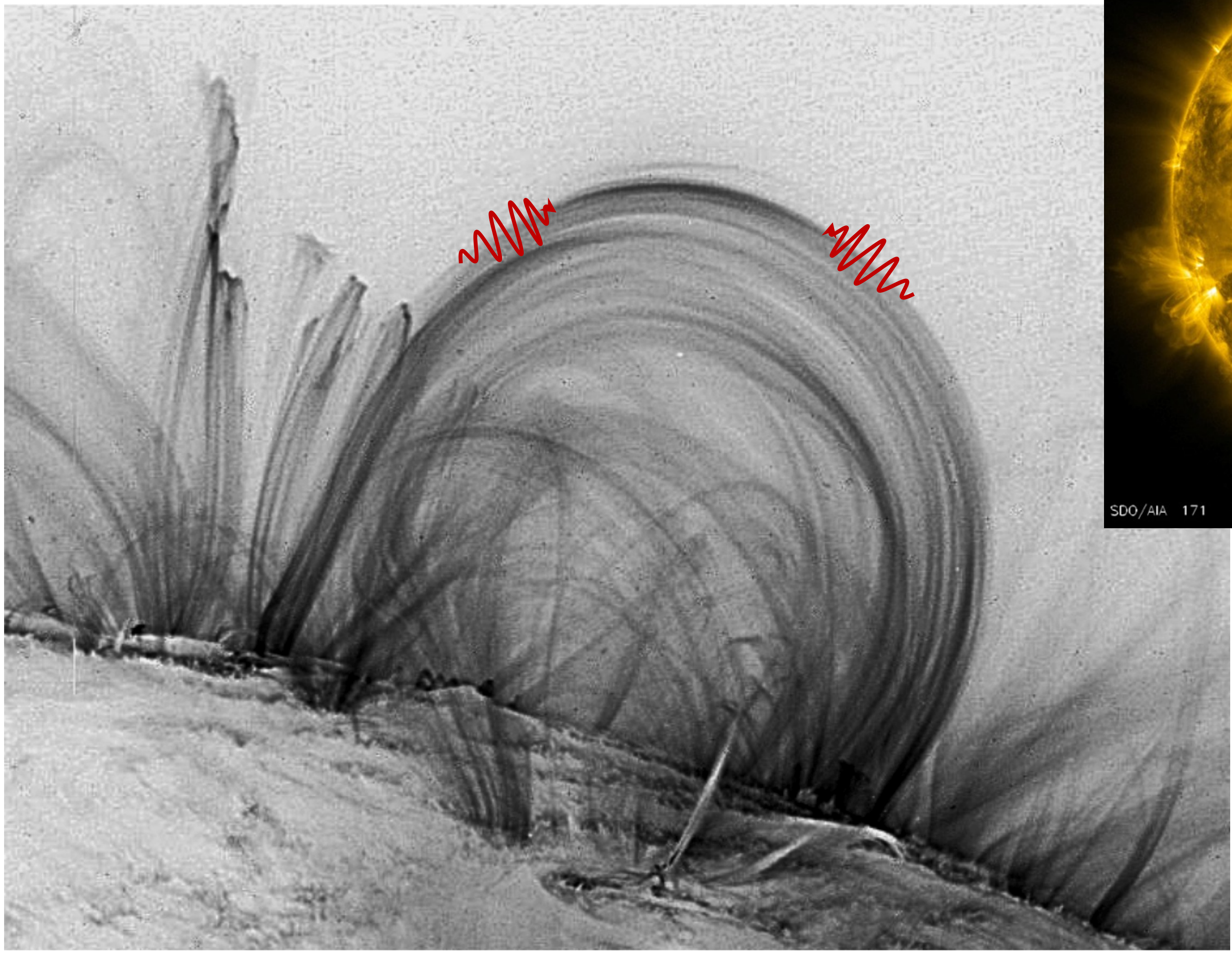
There is a natural asymptotic closure [Benney & Saffman, PRSLA, 1966]

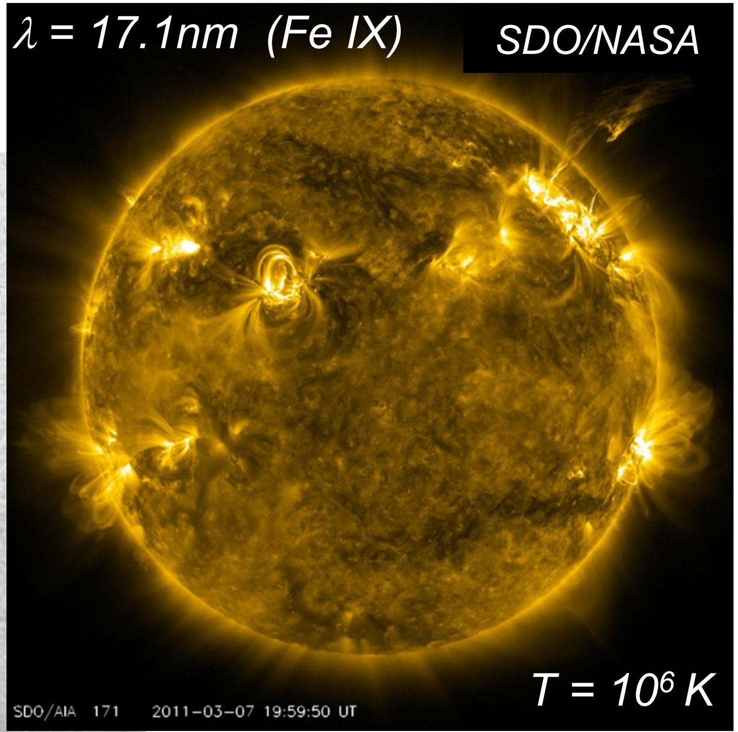
A closed ‘kinetic equation’ can be derived for some second-order (spectral) cumulants (two-point correlation)

The kinetic equation has exact finite flux solutions (called Kolmogorov-Zakharov spectra) which capture the flow of conserved quantities (energy) from sources to sinks

Alfvén wave turbulence

[Rappazzo+, ApJL, 2007; Bigot+, A&A, 2008;
Saur+, A&A, 2002]





TRACE/NASA space telescope (17.1nm); June 1999

Alfvén wave turbulence

$$\frac{\partial \mathbf{z^s}}{\partial t} - s\mathbf{b_0} \cdot \nabla \mathbf{z^s} = -\mathbf{z^{-s}} \cdot \nabla \mathbf{z^s} - \nabla P_* \,,$$

$$\nabla \cdot \mathbf{z^s} = 0\,. \qquad s = \pm$$

$$z_j^s(\mathbf{x}, t) \equiv \int_{\mathbb{R}^3} A_j^s(\mathbf{k}, t)\, e^{i\mathbf{k}\cdot\mathbf{x}}\, d\mathbf{k}\,,$$

$$A_j^s(\mathbf{k}, t) \equiv \epsilon a_j^s(\mathbf{k}, t) e^{-is\omega_k t}\,,$$

$$\frac{\partial a_j^s(\mathbf{k})}{\partial t} = -i\epsilon k_m P_{jn} \int_{\mathbb{R}^6} a_m^{-s}(\mathbf{q})\, a_n^s(\mathbf{p})\, e^{is(\omega_k - \omega_p + \omega_q)t} \underbrace{\delta(\mathbf{k} - \mathbf{p} - \mathbf{q})}_{\text{Triadic interactions}}\, d\mathbf{p} d\mathbf{q}$$

where $P_{jn}(k) \equiv \delta_{jn} - k_j k_n / k^2$

Alfvén wave turbulence

$$\begin{aligned}
\langle a_k^s a_{k'}^{s'}\rangle &= q_{kk'}^{ss'}(\mathbf{k},\mathbf{k}')\delta(\mathbf{k}+\mathbf{k}')\,,\\
\langle a_k^s a_{k'}^{s'} a_{k''}^{s''}\rangle &= q_{kk'k''}^{ss's''}(\mathbf{k},\mathbf{k}',\mathbf{k}'')\delta(\mathbf{k}+\mathbf{k}'+\mathbf{k}'')\,,\\
\langle a_k^s a_{k'}^{s'} a_{k''}^{s''} a_{k'''}^{s'''}\rangle &= q_{kk'k''k'''}^{ss's''s'''}(\mathbf{k},\mathbf{k}',\mathbf{k}'',\mathbf{k}''')\delta(\mathbf{k}+\mathbf{k}'+\mathbf{k}''+\mathbf{k}''')\\
&+ q_{kk'}^{ss'}(\mathbf{k},\mathbf{k}')q_{k''k'''}^{s''s'''}(\mathbf{k}'',\mathbf{k}''')\delta(\mathbf{k}+\mathbf{k}')\delta(\mathbf{k}''+\mathbf{k}''')\\
&+ q_{kk''}^{ss''}(\mathbf{k},\mathbf{k}'')q_{k'k'''}^{s's'''}(\mathbf{k}',\mathbf{k}''')\delta(\mathbf{k}+\mathbf{k}'')\delta(\mathbf{k}'+\mathbf{k}''')\\
&+ q_{kk'''}^{ss'''}(\mathbf{k},\mathbf{k}''')q_{k'k''}^{s's''}(\mathbf{k}',\mathbf{k}'')\delta(\mathbf{k}+\mathbf{k}''')\delta(\mathbf{k}'+\mathbf{k}'')
\end{aligned}$$

$$\begin{aligned}
\frac{\partial\langle a_k^s a_{k'}^{s'}\rangle}{\partial t} &= \left\langle \frac{\partial a_k^s}{\partial t} a_{k'}^{s'}\right\rangle + \left\langle a_k^s \frac{\partial a_{k'}^{s'}}{\partial t}\right\rangle\\
&= i\epsilon \int \sum_{s_p s_q} L_{-kpq}^{-s s_p s_q} \langle a_{k'}^{s'} a_p^{s_p} a_q^{s_q}\rangle e^{i\Omega_{k,pq}t}\delta_{k,pq} d\mathbf{p}d\mathbf{q}\\
&+ i\epsilon \int \sum_{s_p s_q} L_{-k'pq}^{-s' s_p s_q} \langle a_k^{s} a_p^{s_p} a_q^{s_q}\rangle e^{i\Omega_{k',pq}t}\delta_{k',pq} d\mathbf{p}d\mathbf{q}\,.
\end{aligned}$$

Asymptotic closure:
only resonance terms survive

$$\langle A_k^s A_{k'}^{s'}\rangle = \epsilon^2 \langle a_k^s a_{k'}^{s'}\rangle \exp(-i(s\omega_k + s'\omega_{k'})t)$$

$$\Delta(\Omega_{kk'k''}) = \int_0^{t\gg 1/\omega} e^{i\Omega_{kk'k''}t'}dt' = \frac{e^{i\Omega_{kk'k''}t}-1}{i\Omega_{kk'k''}}$$

$$\Delta(x) \to \pi\delta(x) + i\mathcal{P}(1/x)$$

$$\begin{aligned}
\frac{\partial\langle a_k^s a_{k'}^{s'} a_{k''}^{s''}\rangle}{\partial t} &= \left\langle \frac{\partial a_k^s}{\partial t} a_{k'}^{s'} a_{k''}^{s''}\right\rangle + \left\langle a_k^s \frac{\partial a_{k'}^{s'}}{\partial t} a_{k''}^{s''}\right\rangle + \left\langle a_k^s a_{k'}^{s'} \frac{\partial a_{k''}^{s''}}{\partial t}\right\rangle\\
&= i\epsilon \int \sum_{s_p s_q} L_{-kpq}^{-s s_p s_q} \langle a_{k'}^{s'} a_{k''}^{s''} a_p^{s_p} a_q^{s_q}\rangle e^{i\Omega_{k,pq}t}\delta_{k,pq} d\mathbf{p}d\mathbf{q}\\
&+ i\epsilon \int \sum_{s_p s_q} L_{-k'pq}^{-s' s_p s_q} \langle a_k^{s} a_{k''}^{s''} a_p^{s_p} a_q^{s_q}\rangle e^{i\Omega_{k',pq}t}\delta_{k',pq} d\mathbf{p}d\mathbf{q}\\
&+ i\epsilon \int \sum_{s_p s_q} L_{-k''pq}^{-s'' s_p s_q} \langle a_k^{s} a_{k'}^{s'} a_p^{s_p} a_q^{s_q}\rangle e^{i\Omega_{k'',pq}t}\delta_{k'',pq} d\mathbf{p}d\mathbf{q}
\end{aligned}$$

$$\begin{aligned}
&q_{kk'k''}^{ss's''}(\mathbf{k},\mathbf{k}',\mathbf{k}'')\delta(\mathbf{k}+\mathbf{k}'+\mathbf{k}'') = i\epsilon\Delta(\Omega_{kk'k''})\delta(\mathbf{k}+\mathbf{k}'+\mathbf{k}'')\\
&\left\{\left[L_{-k-k'-k''}^{-s-s'-s''} + L_{-k-k''-k'}^{-s-s''-s'}\right] q_{k''-k''}^{s''-s''}(\mathbf{k}'',-\mathbf{k}'')q_{k'-k'}^{s'-s'}(\mathbf{k}',-\mathbf{k}')\right.\\
&+ \left[L_{-k'-k-k''}^{-s'-s-s''} + L_{-k'-k''-k}^{-s'-s''-s}\right] q_{k''-k''}^{s''-s''}(\mathbf{k}'',-\mathbf{k}'')q_{k-k}^{s-s}(\mathbf{k},-\mathbf{k})\\
&+ \left.\left[L_{-k''-k'-k}^{-s''-s'-s} + L_{-k''-k-k'}^{-s''-s-s'}\right] q_{k-k}^{s-s}(\mathbf{k},-\mathbf{k})q_{k'-k'}^{s'-s'}(\mathbf{k}',-\mathbf{k}')\right\}
\end{aligned}$$

Multiple time scale problem
[SG, CUP, 2023]

Alfvén wave turbulence

3-wave interactions

$$\frac{\partial e^s(\mathbf{k})}{\partial t} = \frac{\pi\epsilon^2}{b_0}\int_{\mathbb{R}^6}\frac{(\mathbf{k}_\perp\cdot\mathbf{p}_\perp)^2(\mathbf{k}\times\mathbf{q})_\parallel^2}{k_\perp^2 p_\perp^2 q_\perp^2}\, e^{-s}(\mathbf{q})\,[e^s(\mathbf{p})-e^s(\mathbf{k})]\,\delta(q_\parallel)\,\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})\,d\mathbf{p}d\mathbf{q}$$

where $e^s(\mathbf{k})$ is the energy spectrum associated with shear-Alfvén waves.

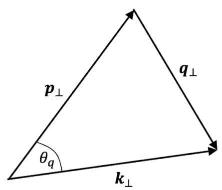
$s = \pm$

$$\mathbf{A}^s(\mathbf{k}) = i\mathbf{k}\times\hat{\mathbf{e}}_\parallel\,\hat{\psi}^s(\mathbf{k}) - \mathbf{k}_\perp k_\parallel\,\hat{\phi}^s(\mathbf{k}) + \hat{\mathbf{e}}_\parallel\,k_\perp^2\,\hat{\phi}^s(\mathbf{k}).$$

Axisymmetric + perpendicular cascade: $2\pi k_\perp e^s(\mathbf{k}) \equiv E^s(k_\perp)f(k_\parallel)$

$$\Longrightarrow \frac{\partial E^s(k_\perp)}{\partial t} = \frac{\epsilon^2 f(0)}{2b_0}\int_{\Delta_\perp}\frac{k_\perp}{q_\perp}(\cos\theta_q)^2\sin\theta_p\,E^{-s}(q_\perp)\,[k_\perp E^s(p_\perp)-p_\perp E^s(k_\perp)]\,dp_\perp dq_\perp$$

kinetic equation



Alfvén wave turbulence depends on the slow mode ($k_{//} = 0$)

Kinetic equation: exact solutions

$$E^{\pm}(k_\perp) \sim k_\perp^{n_\pm} \implies \boxed{n_+ + n_- = -4}$$

locality condition
$-3 < n_\pm < -1$

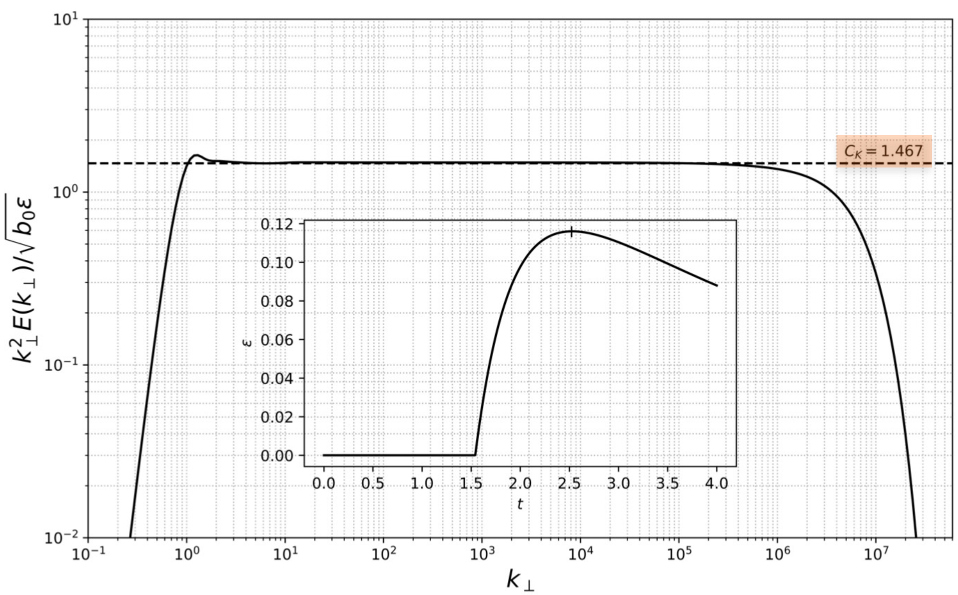
For balanced turbulence: $n_+ = n_-$

$$\boxed{E(k_\perp) = C_K \sqrt{b_0 \varepsilon} k_\perp^{-2}}. \qquad C_K \simeq 1.467$$

Kolmogorov-Zakharov spectrum

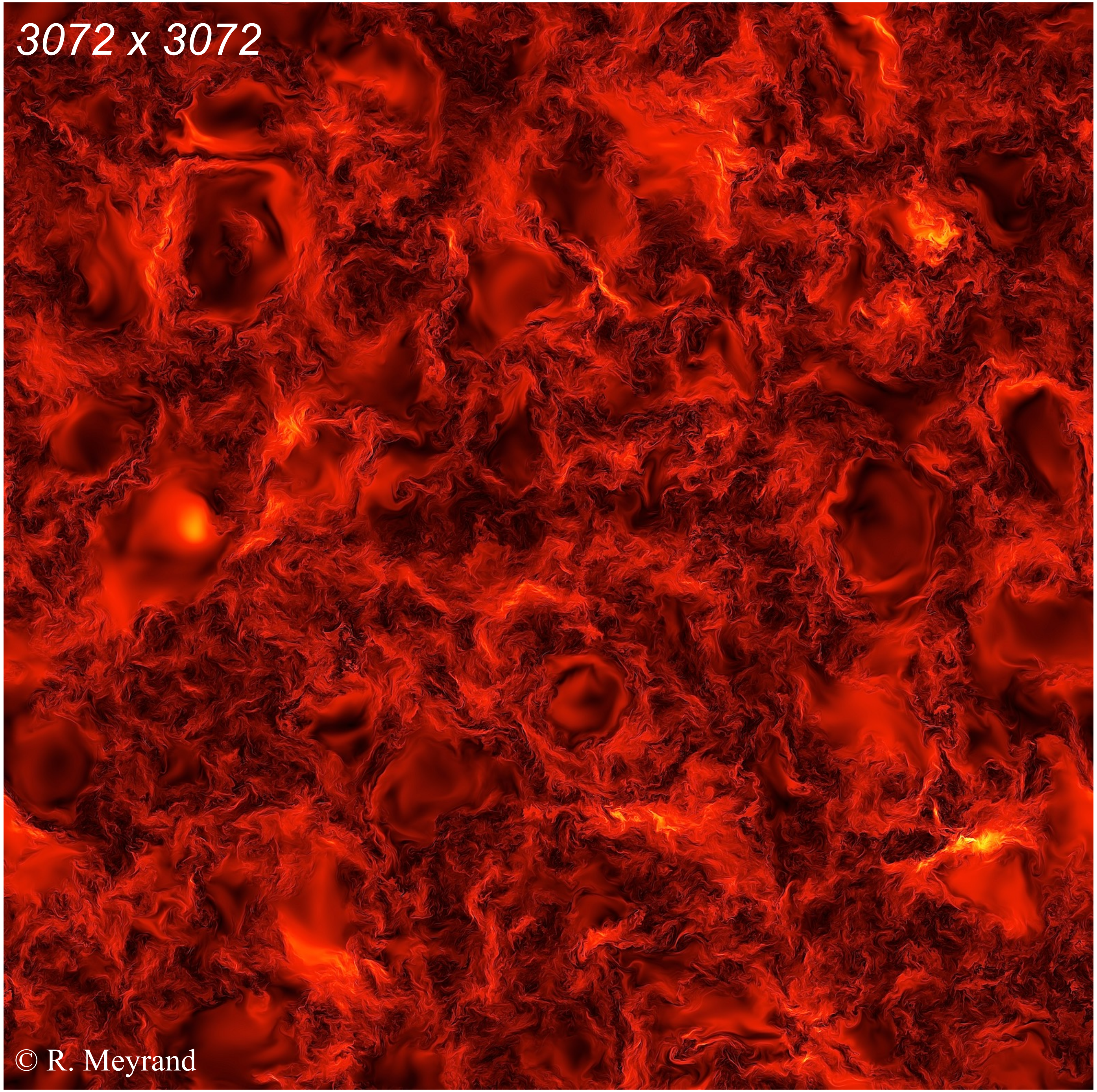
[SG+, JPP, 2000]

Direct cascade can be proved (positive energy flux ε)



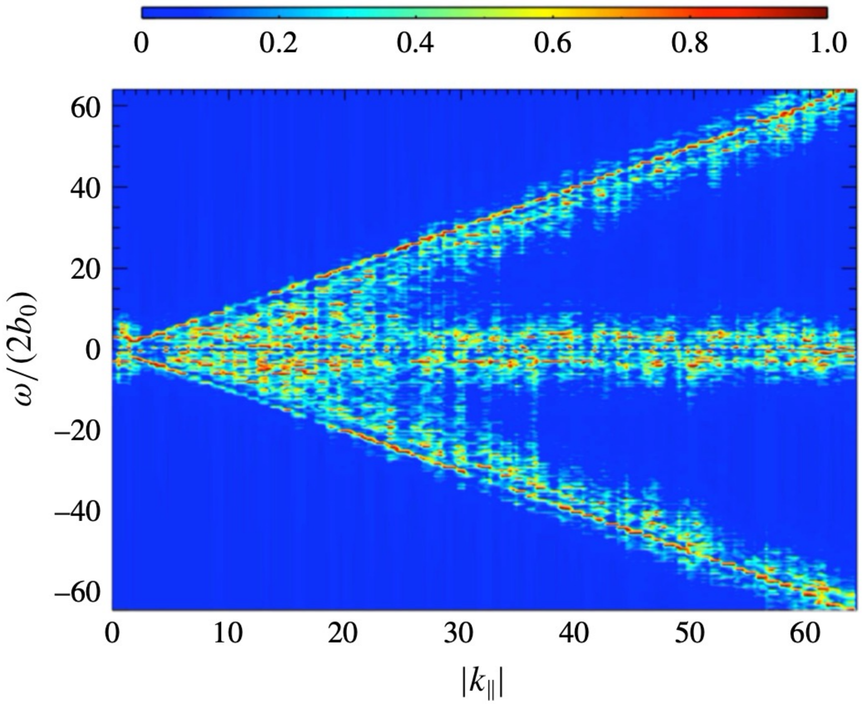
Simulation of the kinetic equation

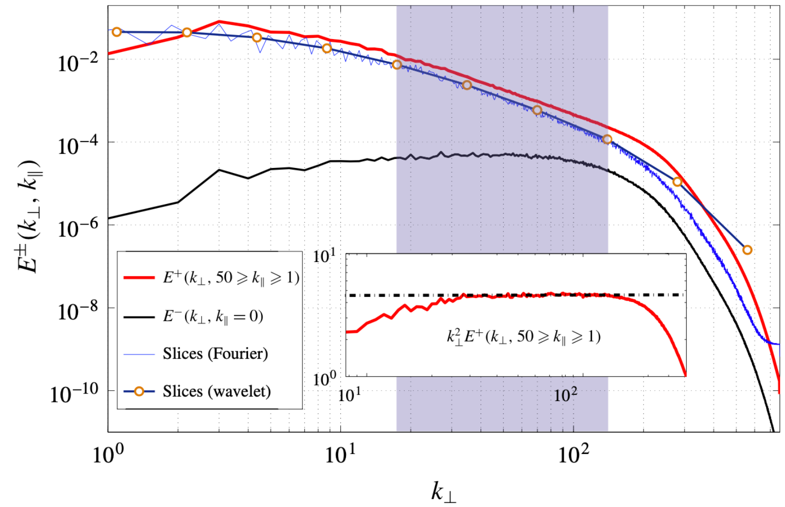
DNS of Alfvén wave turbulence



DNS of Alfvén wave turbulence

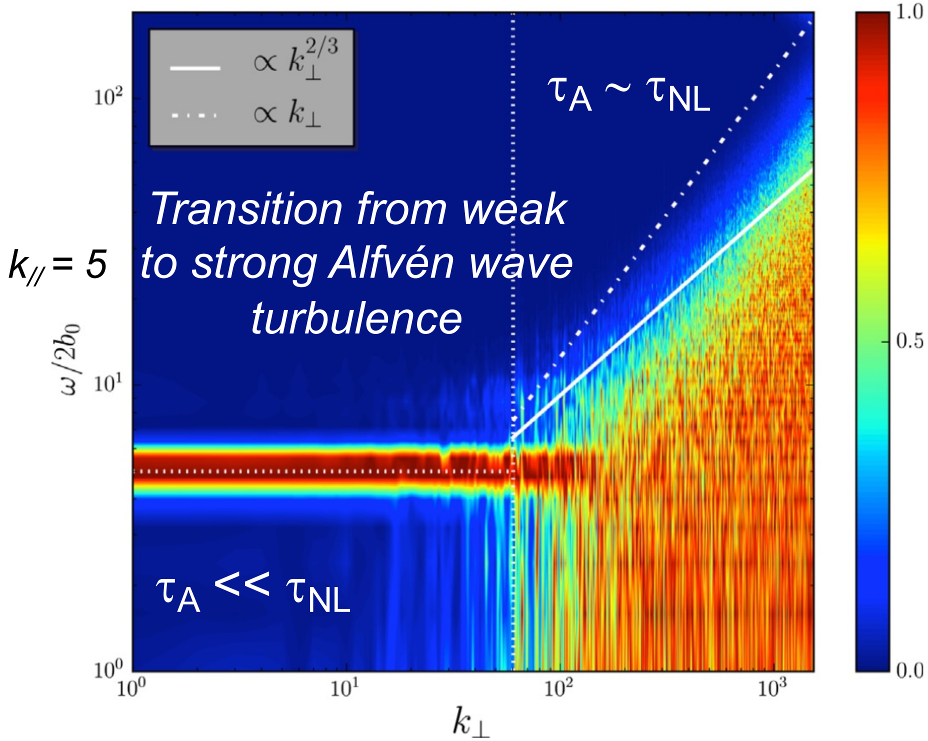
[Meyrand+, JFMR, 2015]





$\omega - k$ *energy spectra*

[Meyrand+, PRL, 2016]

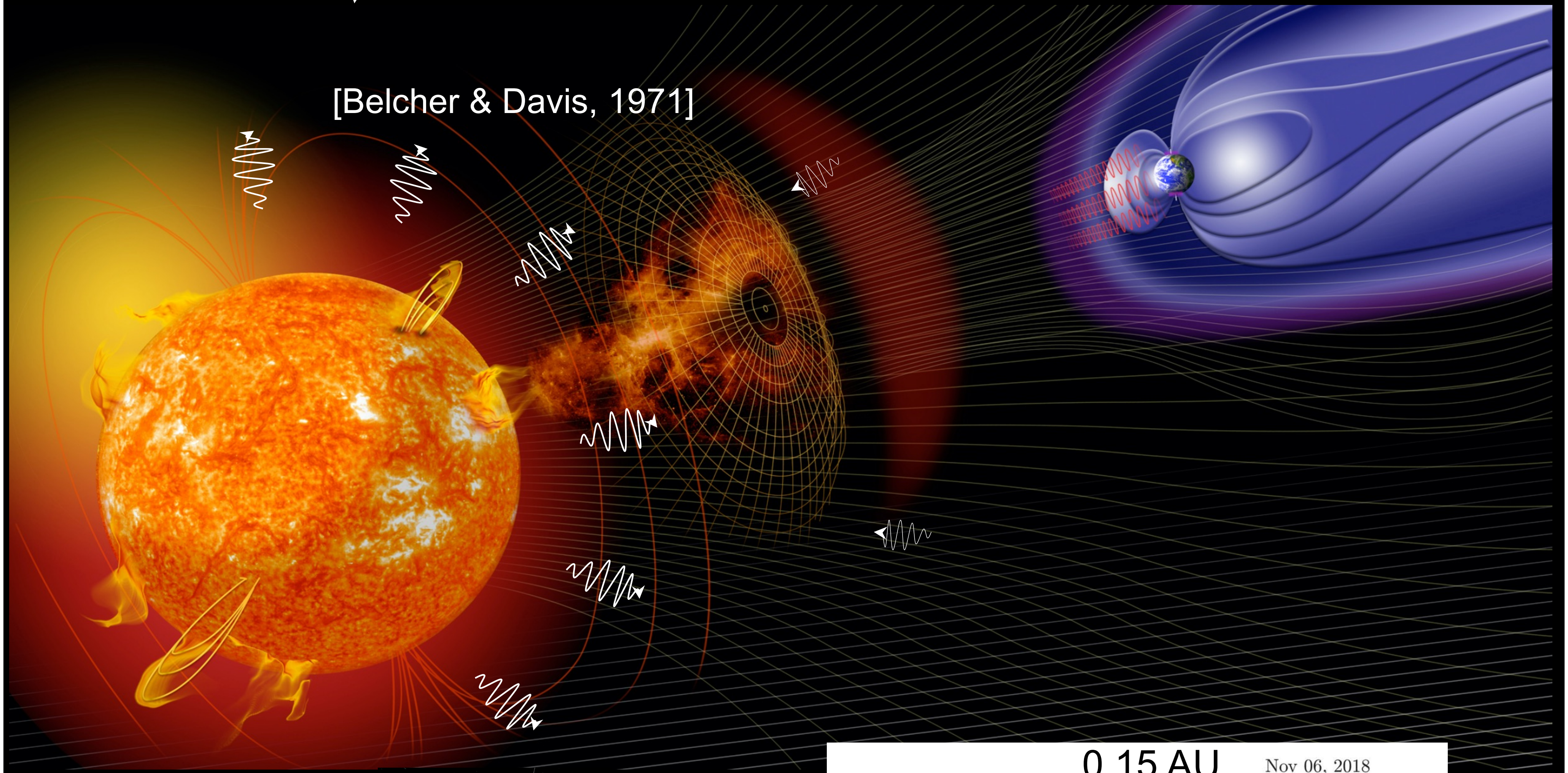


Direct evidence of Alfvén wave turbulence

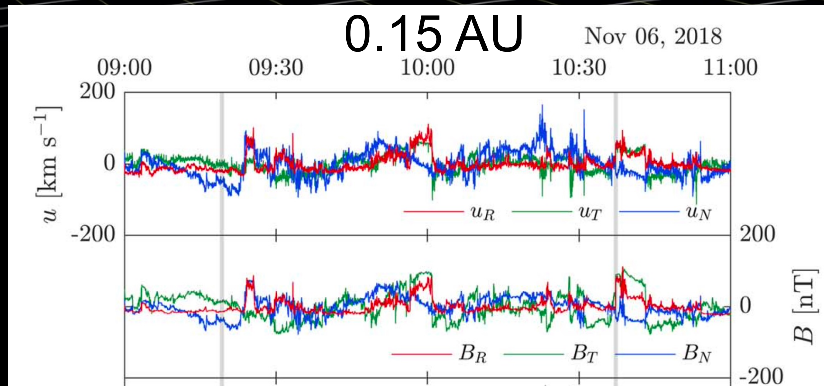
Below MHD scales

[Parker, 1958]

Solar wind, turbulence and waves

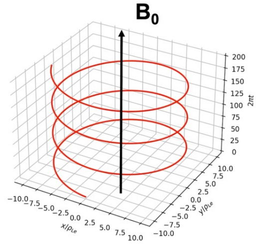


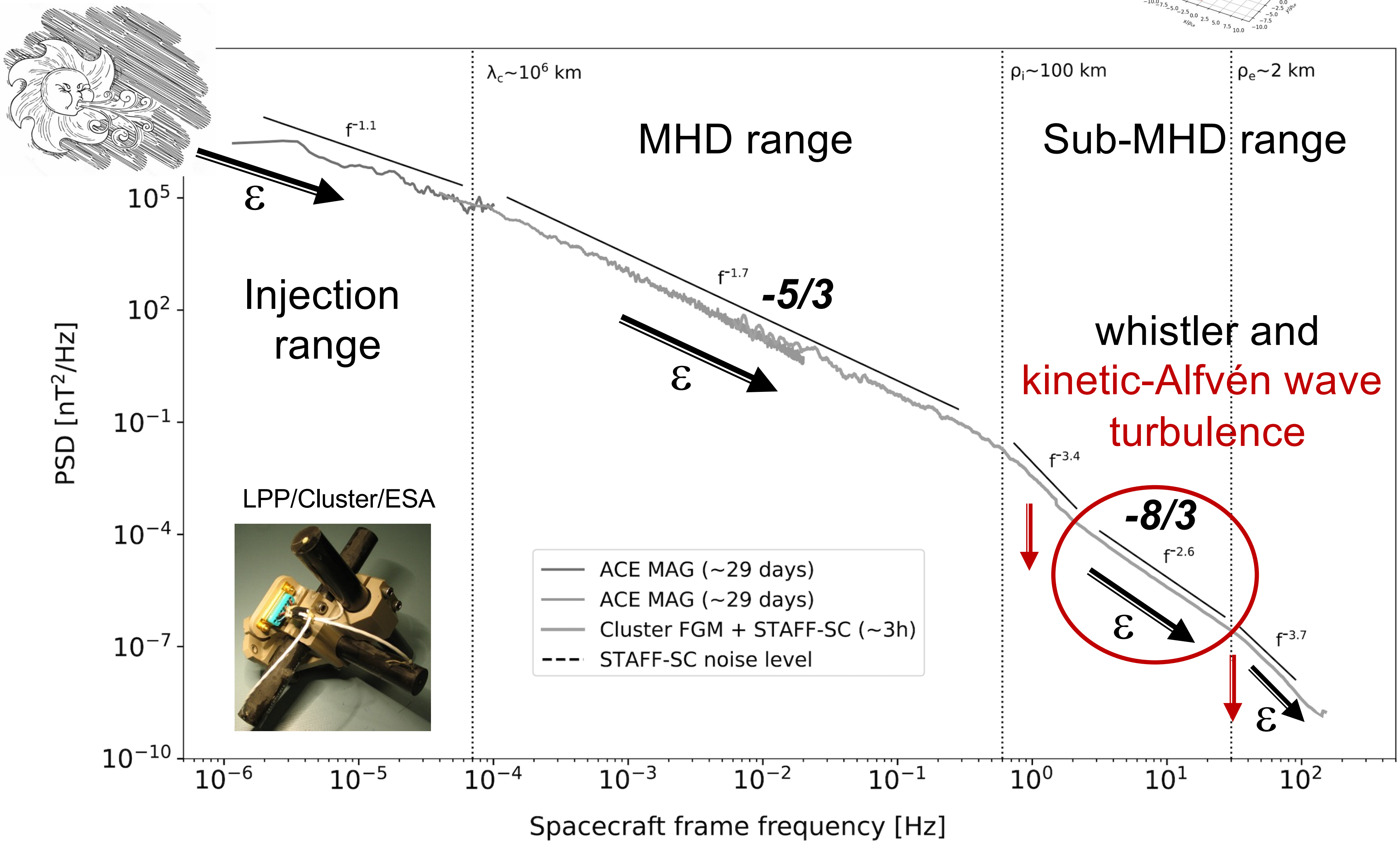
PSP/NASA



Solar wind turbulence

At 1 AU





[Kiyani+, PTRSA, 2015]

In Lecture 1

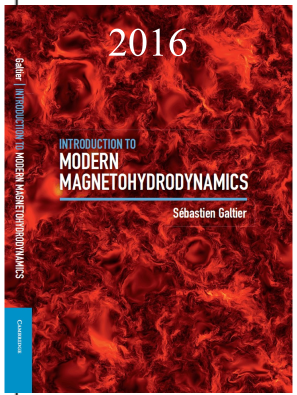
- Hall MHD equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0\,, \qquad \mathrm{P} = \mathrm{K}n^{\gamma}$$

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla P + \frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B} + \tilde{\nu}\Delta\mathbf{u} + \frac{\tilde{\nu}}{3}\nabla(\nabla\cdot\mathbf{u})\,,$$

$$\frac{\partial \mathbf{B}}{\partial t} = \underbrace{\nabla\times(\mathbf{u}\times\mathbf{B})}_{\text{ideal standard MHD}} - \underbrace{\nabla\times\left(\frac{(\nabla\times\mathbf{B})\times\mathbf{B}}{\mu_0 n e}\right)}_{\text{Hall effect}} + \underbrace{\eta\Delta\mathbf{B}}_{\text{collisions}}\,,$$

$$\nabla\cdot\mathbf{B} = 0\,.$$



$\tilde{\nu}$ is the dynamic viscosity

η is the **magnetic diffusivity**

$$P_* \equiv P/\rho_0 + b^2/2$$

Hall effect

ion inertial length

Incompressible case ⟹

$$\mathbf{b} \equiv \mathbf{B}/\sqrt{\mu_0\rho_0}$$

$$\nabla\cdot\mathbf{u} = 0\,,$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla P_* + \mathbf{b}\cdot\nabla\mathbf{b} + \nu\Delta\mathbf{u}\,,$$

$$\frac{\partial \mathbf{b}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{b} = \mathbf{b}\cdot\nabla\mathbf{u} - d_i\nabla\times((\nabla\times\mathbf{b})\times\mathbf{b}) + \eta\Delta\mathbf{b}\,,$$

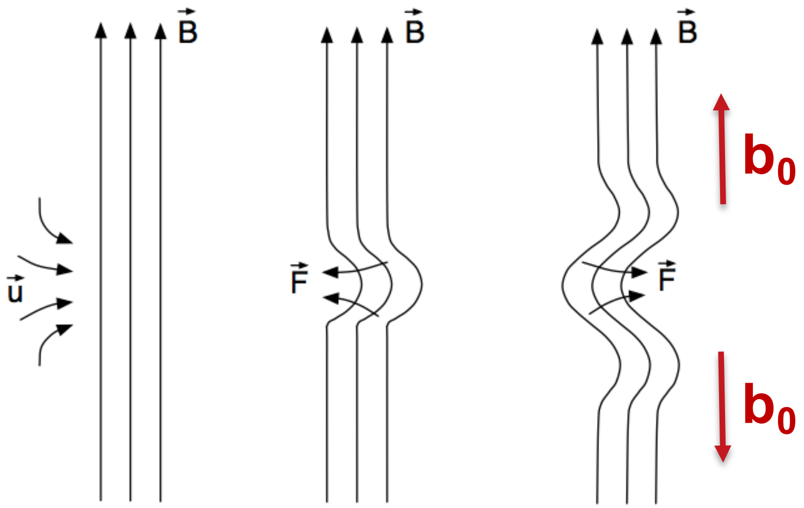
$$\nabla\cdot\mathbf{b} = 0\,,$$

$$\omega_{pi} = \sqrt{\frac{n_i e^2}{m_i \epsilon_0}}$$

$$d_i = c/\omega_{pi}$$

$d_i \approx 100\,\mathrm{km}$ in the solar wind ($f > 1\,\mathrm{Hz}$) at 1 AU

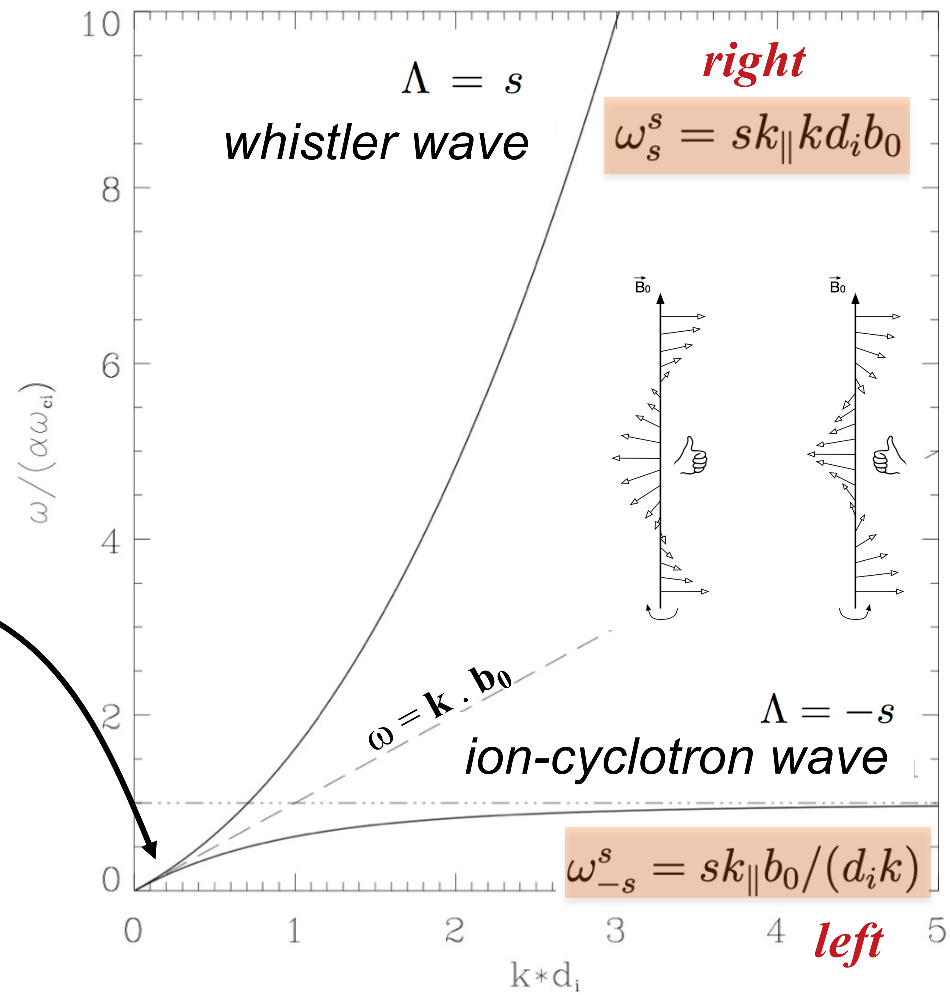
Incompressible Hall MHD waves



Alfvén wave

$$\omega^2 = (b_0 k_\parallel)^2 = (\mathbf{b}_0 \cdot \mathbf{k})^2 \, .$$

$$\omega_{ci} \equiv b_0/d_i \qquad \alpha \equiv k_\parallel/k$$



$$\omega_\Lambda^s = \frac{s k_\parallel k d_i b_0}{2}\left(s\Lambda + \sqrt{1 + \frac{4}{d_i^2 k^2}}\right) ,$$

s=±1 and Λ=±1

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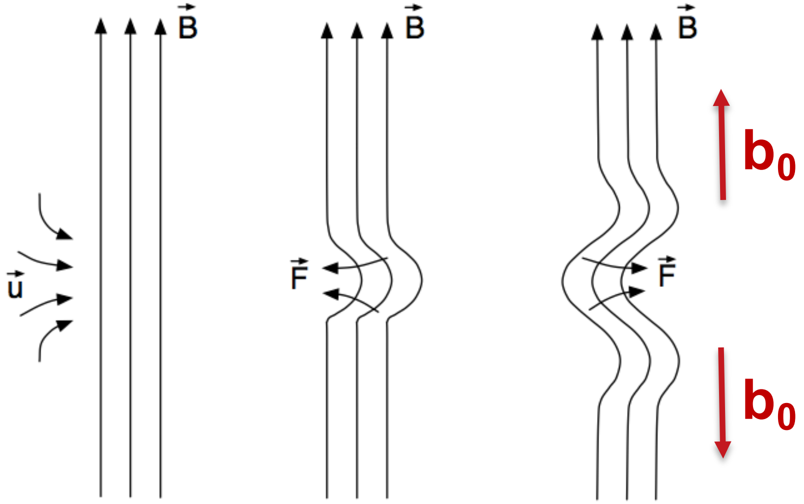

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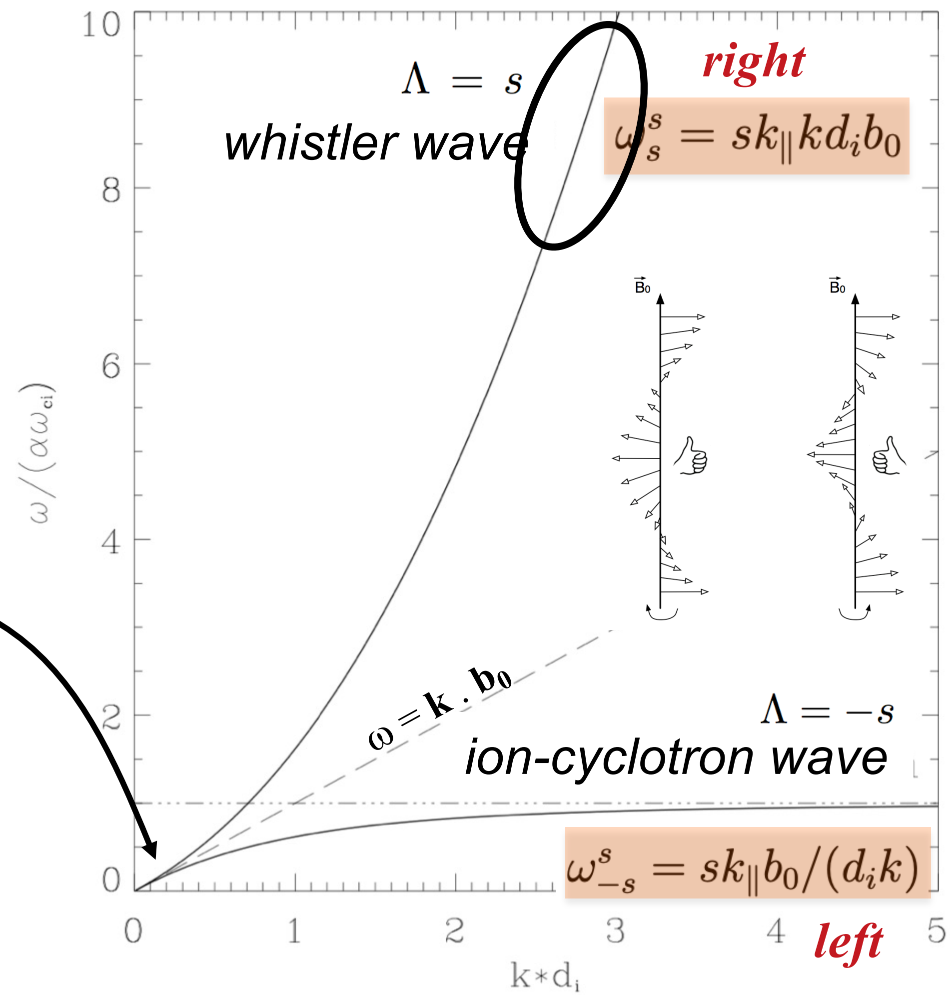

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s=±1 and Λ=±1

Electron MHD (EMHD)

$$\nabla \cdot \mathbf{u} = 0\,,$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P_* + \mathbf{b} \cdot \nabla \mathbf{b} + \nu \Delta \mathbf{u}\,,$$

$$\frac{\partial \mathbf{b}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{b} = \mathbf{b} \cdot \nabla \mathbf{u} - d_i \nabla \times ((\nabla \times \mathbf{b}) \times \mathbf{b}) + \eta \Delta \mathbf{b}\,,$$

$$\nabla \cdot \mathbf{b} = 0\,,$$

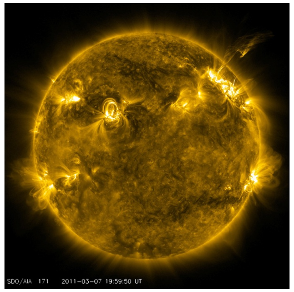
$\Downarrow$ Small-scale limit $\ell \ll d_i$

$$\frac{\partial \mathbf{b}}{\partial t} = -d_i \nabla \times ((\nabla \times \mathbf{b}) \times \mathbf{b}) + \eta \Delta \mathbf{b}$$

- Ions are too heavy to follow electrons ($\mathbf{u} \approx \mathbf{u}_i = \mathbf{0}$)
- Two ideal invariants: E^b direct cascade of energy
  H^b inverse cascade of helicity

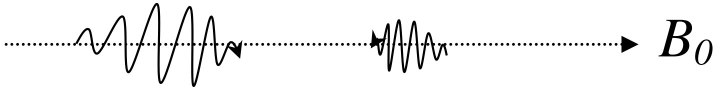
Helicity barrier

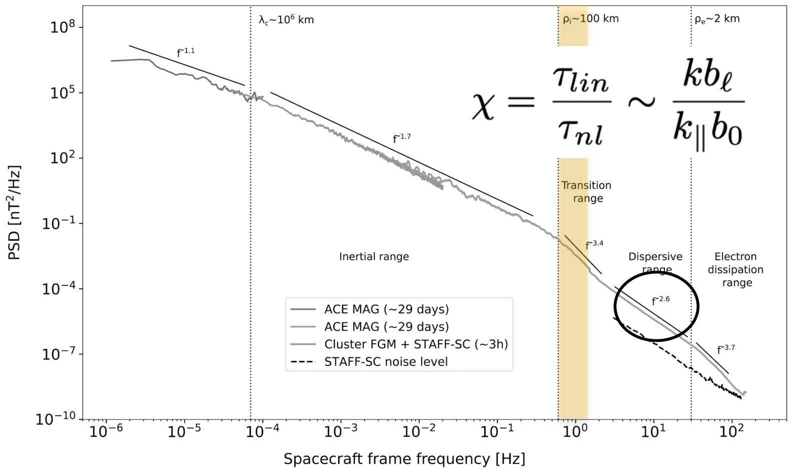
[Meyrand+, JPP, 2021; Squire+, Nat. Astron., 2022]

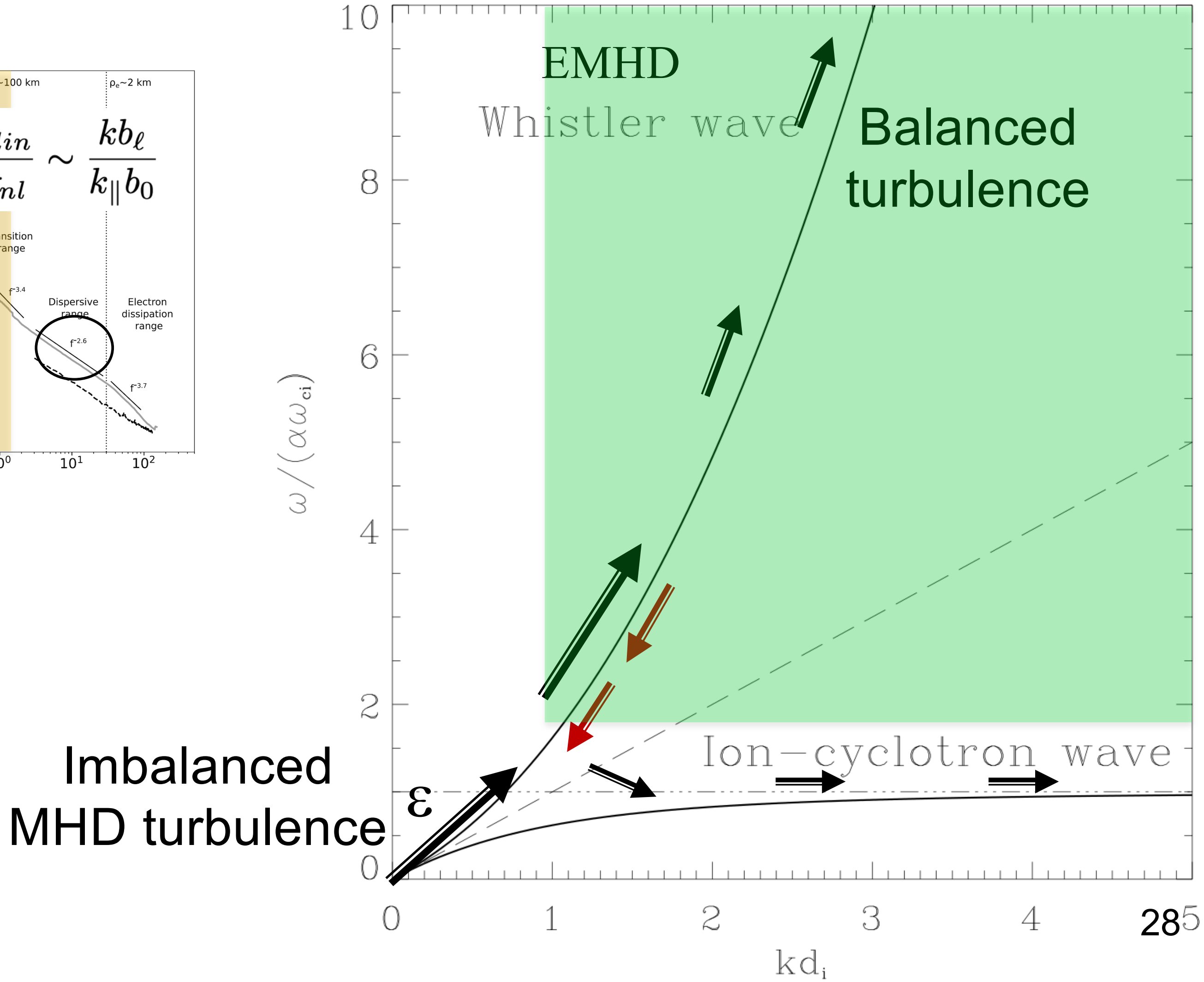


Imbalanced turbulence => $H^b \neq 0$

(for balanced turbulence, $H^b = 0$)

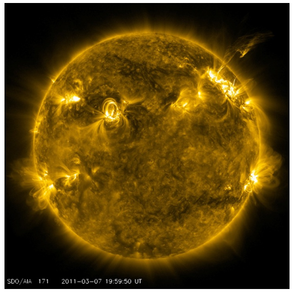






Helicity barrier

[Meyrand+, JPP, 2021; Squire+, Nat. Astron., 2022]



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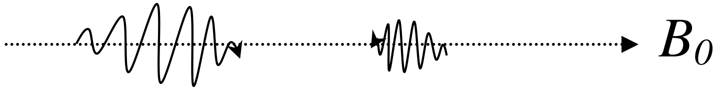
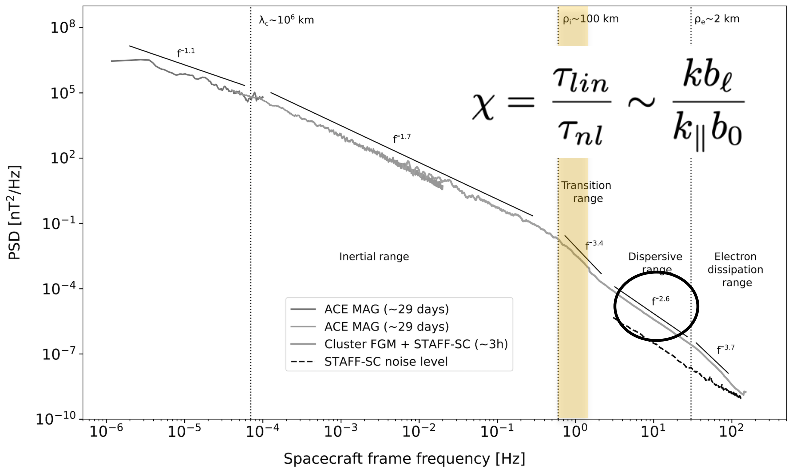
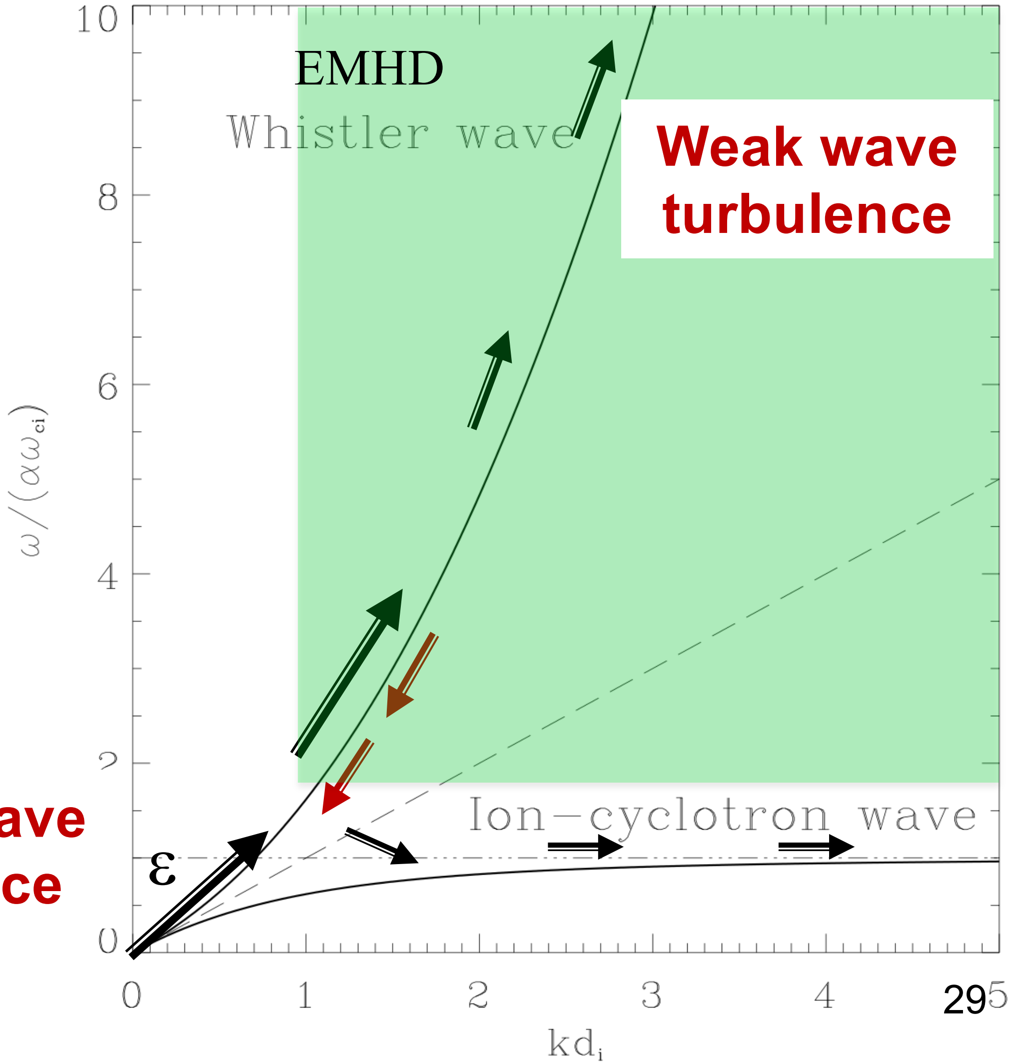
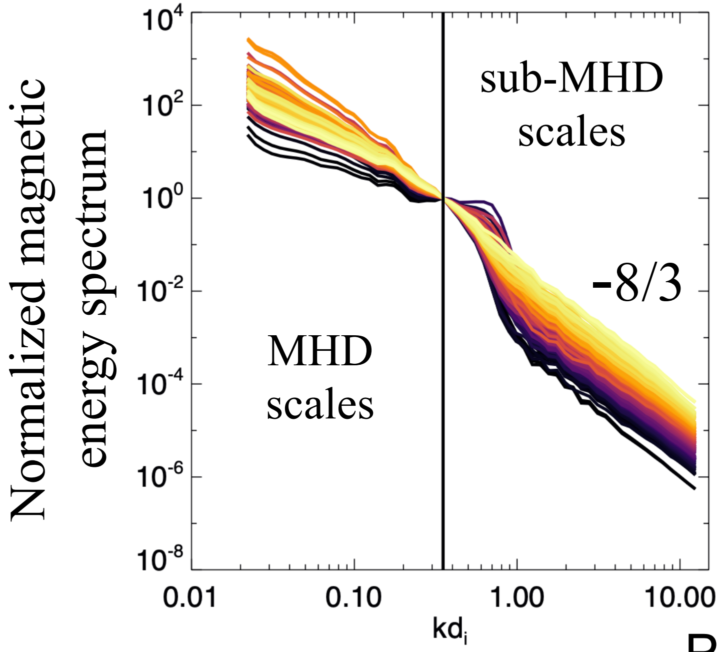


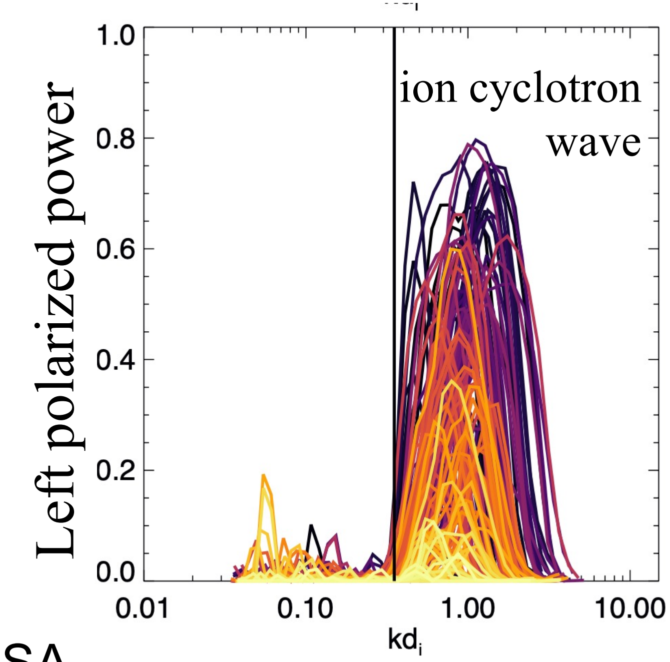


Strong wave turbulence



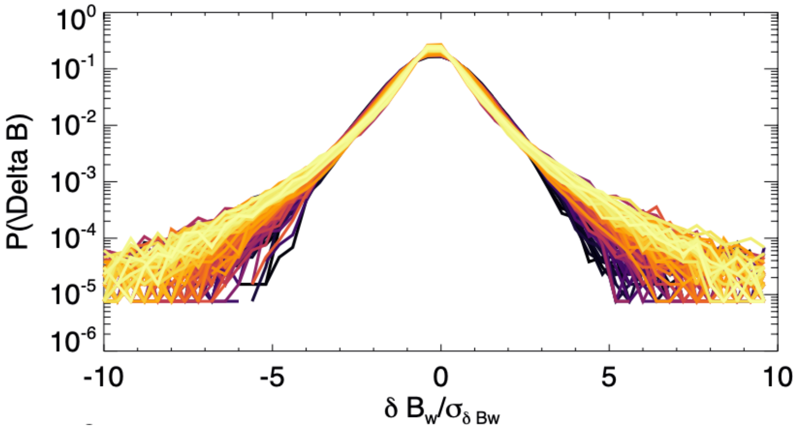
Wave turbulence in the solar wind

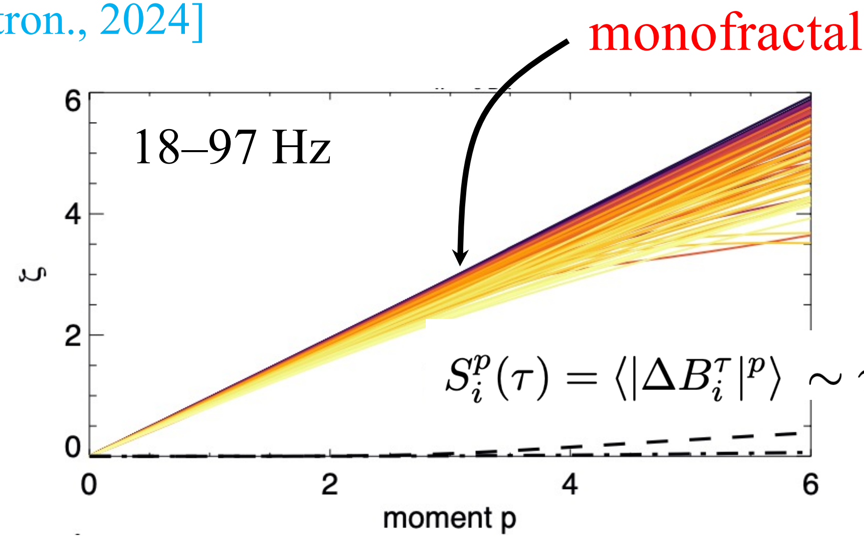




PSP/NASA

[Bowen+, Nat. Astron., 2024]





$$S_i^p(\tau) = \langle |\Delta B_i^\tau|^p \rangle \sim \tau^{\zeta(p)}$$

$$\Delta B_i^\tau = B_i(t) - B_i(t+\tau)$$

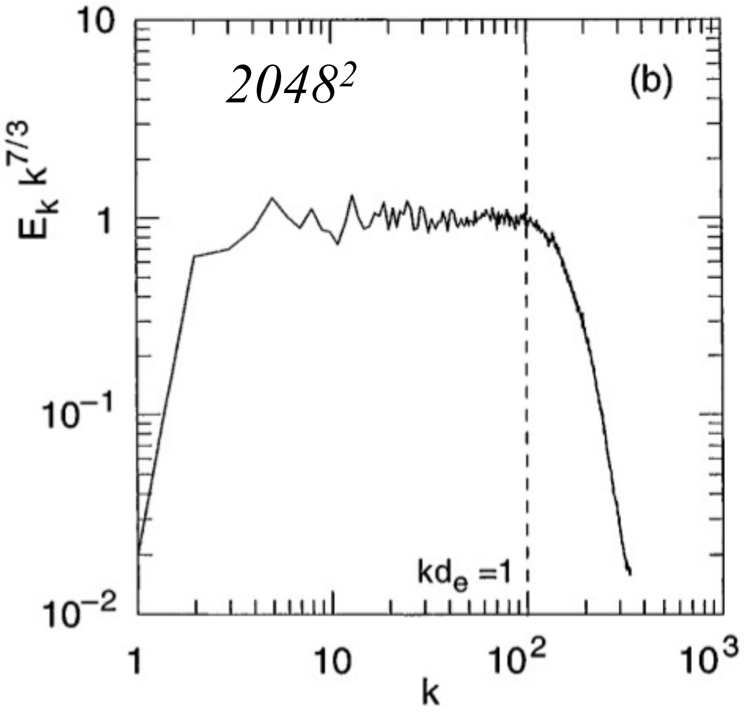
Strong EMHD turbulence

$$\frac{\partial \mathbf{b}}{\partial t} = -d_i \nabla \times ((\nabla \times \mathbf{b}) \times \mathbf{b}) + \eta \Delta \mathbf{b} \qquad \nabla \cdot \mathbf{b} = 0,$$

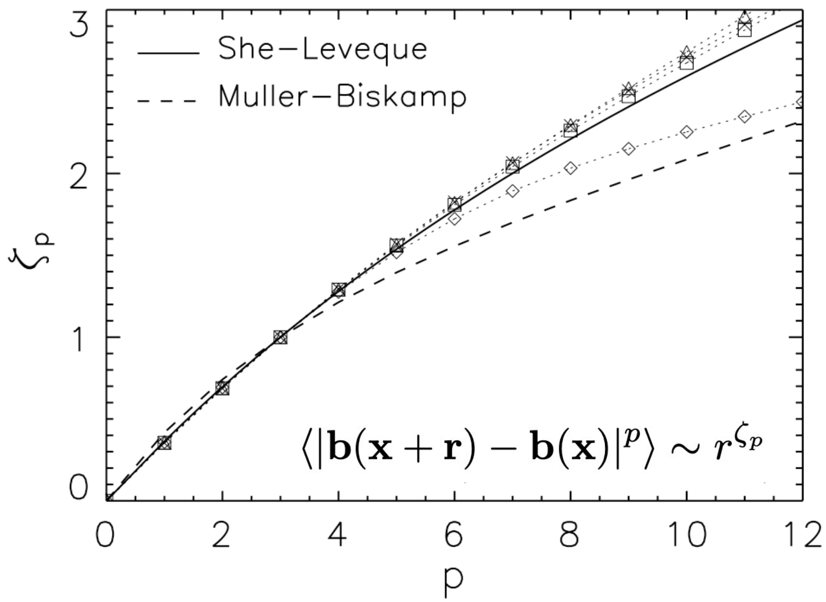
Phenomenology: $\tau_{nl} \sim \ell^2/(d_i b_\ell)$ $\qquad \varepsilon \sim \frac{E(k)k}{\tau_{tr}} \sim d_i k^{7/2} E^{3/2}(k)$

$$\Rightarrow \boxed{E^b(k) \sim (\varepsilon/d_i)^{2/3} k^{-7/3}}$$ 1D isotropic spectrum

[Biskamp+, PRL, 1996]



[Cho & Lazarian, ApJ, 2009; SG, PRE, 2008]



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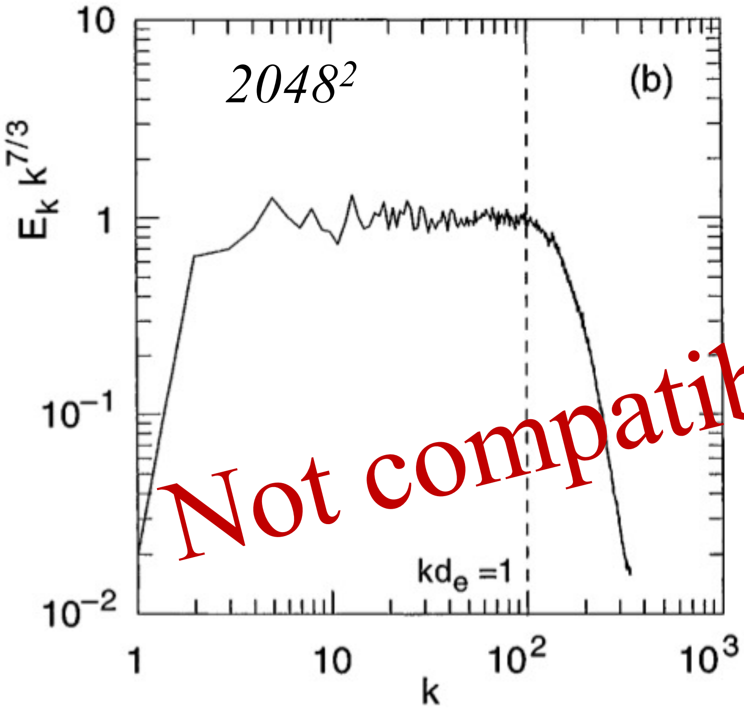

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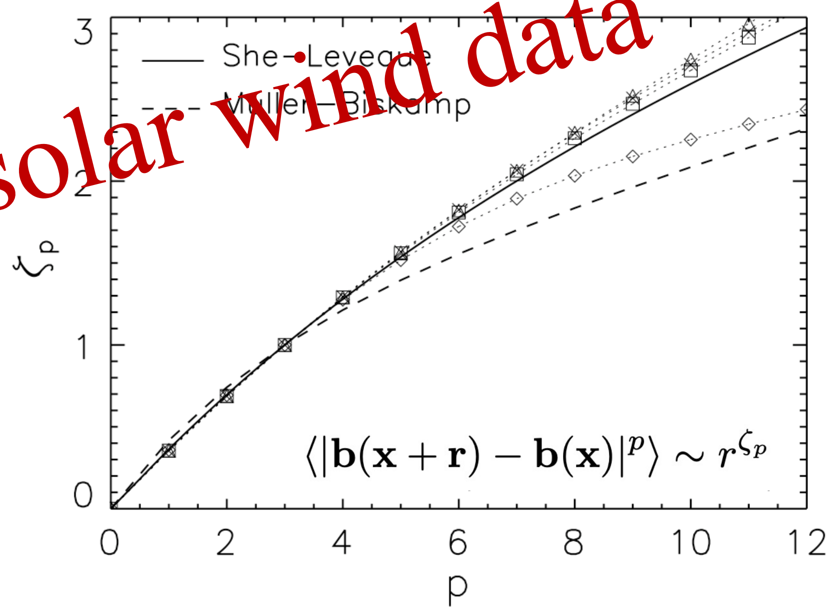


Not compatible with solar wind data

Weak EMHD turbulence

$$\partial_t \mathbf{b} + d_i b_0 \partial_\| (\nabla \times \mathbf{b}) = -d_i \nabla \times [(\nabla \times \mathbf{b}) \times \mathbf{b}], \qquad \nabla \cdot \mathbf{b} = 0,$$

$$|\mathbf{b}| \ll |\mathbf{b_0}|$$

- Fourier transform
- Use a complex helicity basis: $\mathbf{h}^s(\mathbf{k}) \equiv \mathbf{h}_\mathbf{k}^s = (\hat{\mathbf{e}}_k \times \hat{\mathbf{e}}_\|) \times \hat{\mathbf{e}}_k + is(\hat{\mathbf{e}}_k \times \hat{\mathbf{e}}_\|),$

$$is(\hat{\mathbf{e}}_k \times \mathbf{h}_\mathbf{k}^s) = \mathbf{h}_\mathbf{k}^s, \quad \mathbf{k} \cdot \mathbf{h}_\mathbf{k}^s = 0, \quad \mathbf{h}_\mathbf{k}^s \cdot \mathbf{h}_\mathbf{k}^s = 0$$

$$\mathbf{b}_\mathbf{k} = \Sigma_s a^s(\mathbf{k}) e^{-is\omega_k t} \mathbf{h}_\mathbf{k}^s \equiv \Sigma_s a_\mathbf{k}^s e^{-is\omega_k t} \mathbf{h}_\mathbf{k}^s,$$

Linear solution: $\omega = d_i b_0 k_\| k$ (helical wave)

$$\Longrightarrow \quad \partial_t a_\mathbf{k}^s = \Sigma_{s_p s_q} \int L_{-\mathbf{kpq}}^{s s_p s_q} a_\mathbf{p}^{s_p} a_\mathbf{q}^{s_q} e^{i g_{k,pq} t} \delta_{k,pq} \, d_{pq},$$

three-wave interactions

$$L_{\mathbf{kpq}}^{s s_p s_q} = \left(\frac{isk^3}{2k_\perp^2} \right) [(\mathbf{q} \cdot \mathbf{h}_\mathbf{k}^s)(\mathbf{h}_\mathbf{p}^{s_p} \cdot \mathbf{h}_\mathbf{q}^{s_q}) - (\mathbf{q} \cdot \mathbf{h}_\mathbf{p}^{s_p})(\mathbf{h}_\mathbf{k}^s \cdot \mathbf{h}_\mathbf{q}^{s_q})].$$

$$g_{k,pq} = s\omega_k - s_p \omega_p - s_q \omega_q$$

Wave turbulence in EMHD

$k_\perp \gg |k_\|| > 0$

E_k, H_k: axisymmetric spectra

$$\partial_t \begin{Bmatrix} E_k \\ H_k \end{Bmatrix} = \frac{\epsilon^2}{16} \sum_{s s_p s_q} \int \frac{s_p\, p_\perp k_\| p_\|}{q_\perp} \left(\frac{s_q q_\perp - s_p p_\perp}{k_\|} \right)^2 (s k_\perp + s_p p_\perp + s_q q_\perp)^2 \sin\theta_q$$

$$\begin{Bmatrix} s k_\perp \left[E_q (p_\perp E_k - k_\perp E_p)/(k_\perp p_\perp q_\perp) + s_q\, H_q\, (s H_k - s_p H_p) \right] \\ E_q (s\, H_k - s_p H_p)/q_\perp + s_q\, H_q (p_\perp E_k - k_\perp E_p)/(k_\perp p_\perp) \end{Bmatrix}$$

$$\delta(k_\| + p_\| + q_\|)\, \delta(s k_\perp k_\| + s_p p_\perp p_\| + s_q q_\perp q_\|)\, dp_\perp dq_\perp dp_\| dq_\| \,.$$

[SG & Batthacharjee, PoP, 2003]

$H_k = 0$

$E_k \sim k_\perp^n |k_\||^m$

Zakharov-Kuznetsov transformation →

Kolmogorov-Zakharov spectrum

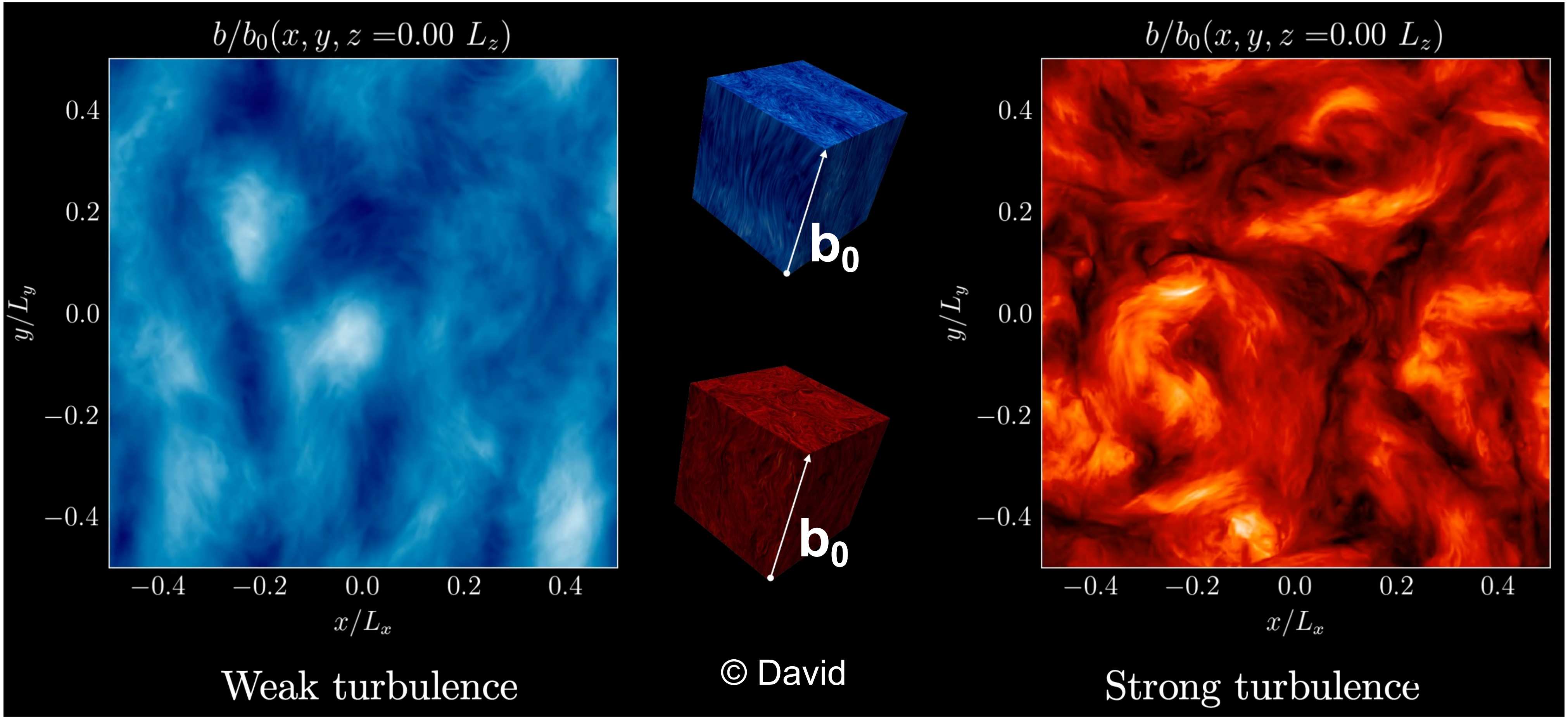
$n = -5/2,\ m = -1/2$

Direct cascade of energy

Direct numerical simulation

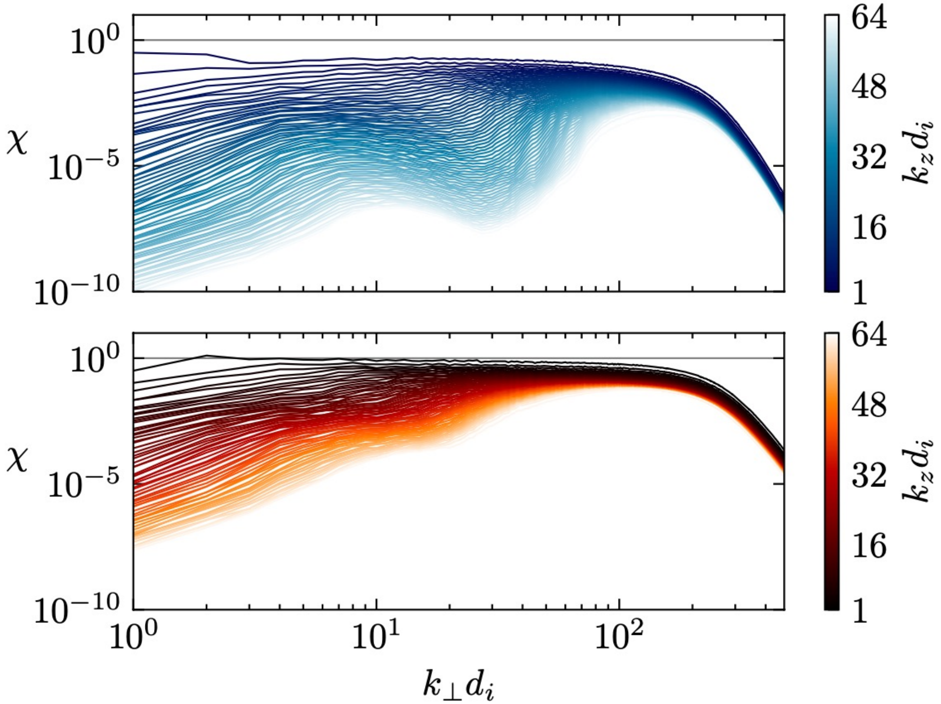
- AsteriX (pseudo-spectral) code – © Meyrand
- Resolution: 1024x1024x128; b_0=1
- Hyperdissipation; forcing at $1.5 < k_\perp < 2.5$ $|k_z| = 1$
- No magnetic helicity injection

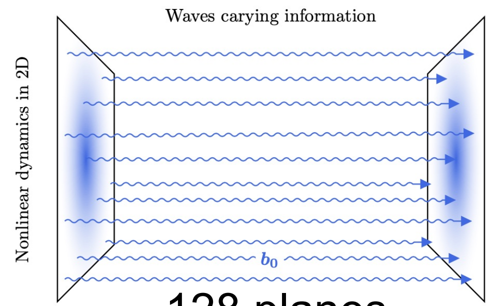
Kinetic-Alfvén wave turbulence



Direct numerical simulation

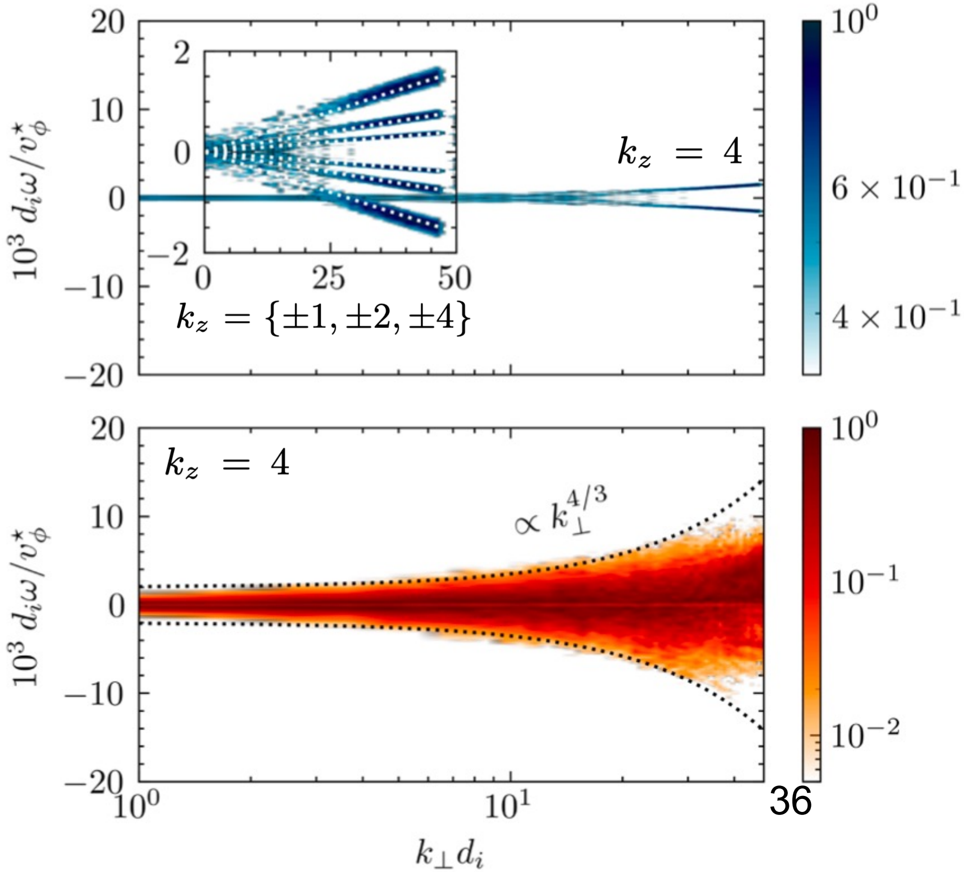
[David+, PRL, Wednesday]





128 planes

$$\chi\left(k_\perp, k_\parallel\right) \equiv \frac{\tau_{\rm lin}}{\tau_{\rm nl}} \simeq \sqrt{\frac{2k_\perp^3 E\left(k_\perp, k_\parallel\right)}{k_\parallel b_0^2}},$$



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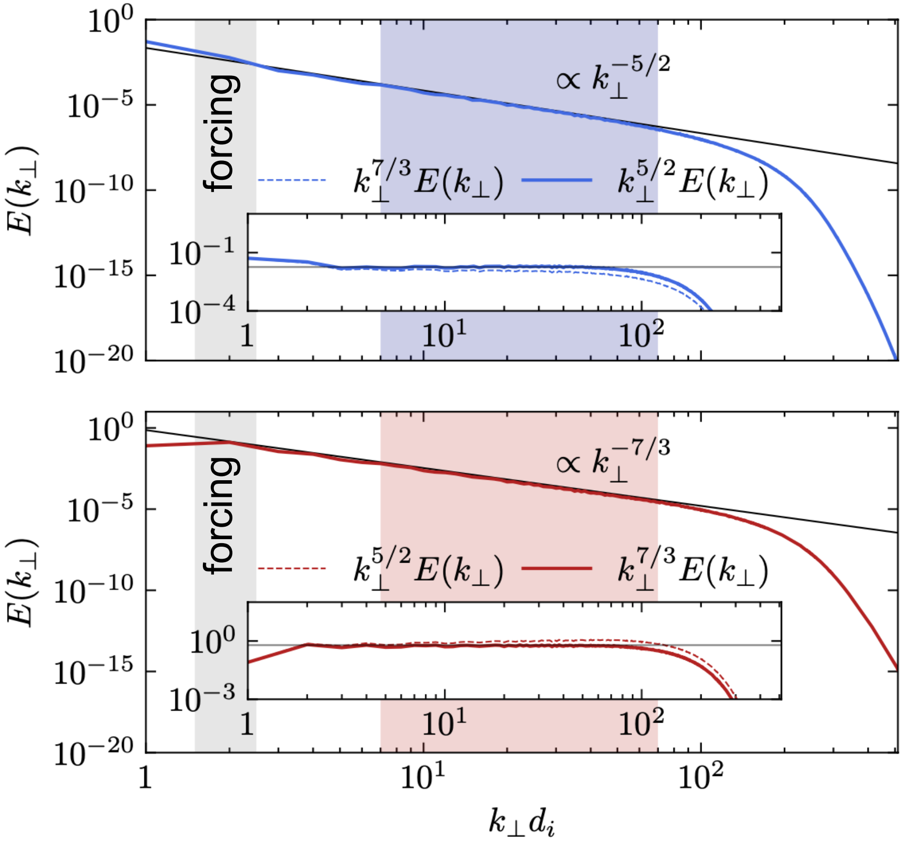
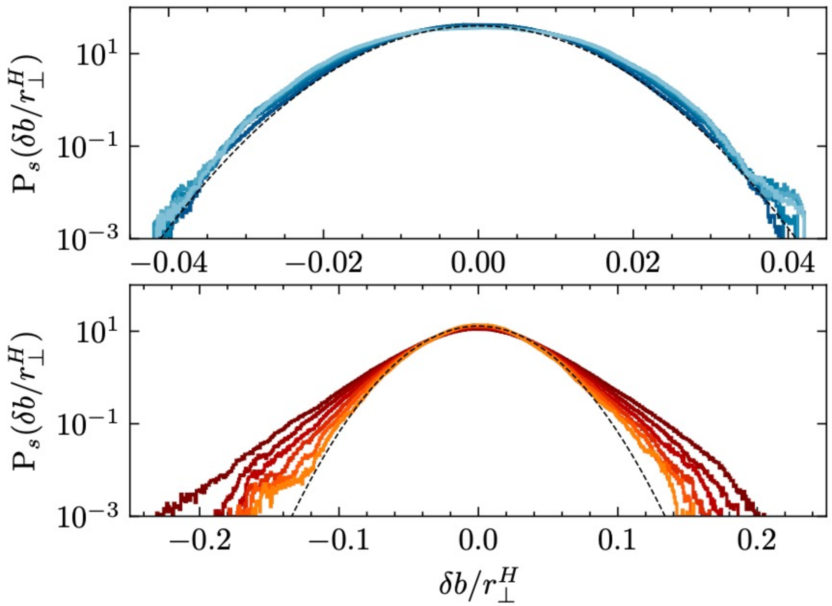
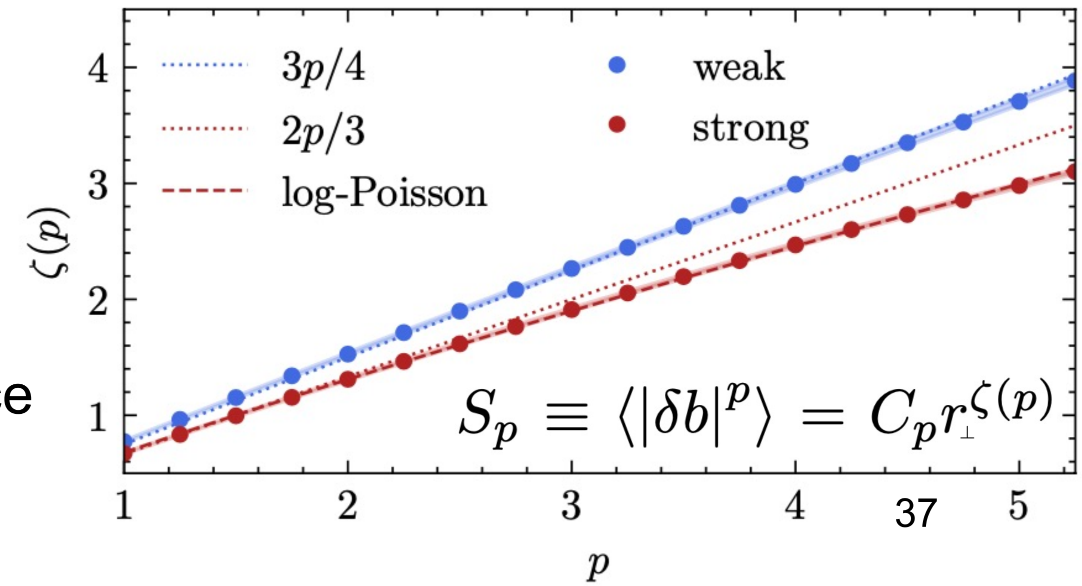




[David+, PRL, Wednesday]



Weak Kinetic-Alfvén wave turbulence
is monofractal

$\zeta(p) = 3p/4$

Super-local wave turbulence

$$0 < \epsilon_p \sim \epsilon_q \ll 1$$

$$p_\perp = k_\perp(1+\epsilon_p)$$

$$q_\perp = k_\perp(1+\epsilon_q)$$

Kinetic equation $\xrightarrow[\text{Taylor expansion}]{}$ nonlinear diffusion equation

$$\frac{\partial E(k_\perp)}{\partial t} = C\frac{\partial}{\partial k_\perp}\left[k_\perp^7 E(k_\perp)\frac{\partial(E(k_\perp)/k_\perp)}{\partial k_\perp}\right] - \eta k_\perp^6 E(k_\perp)$$

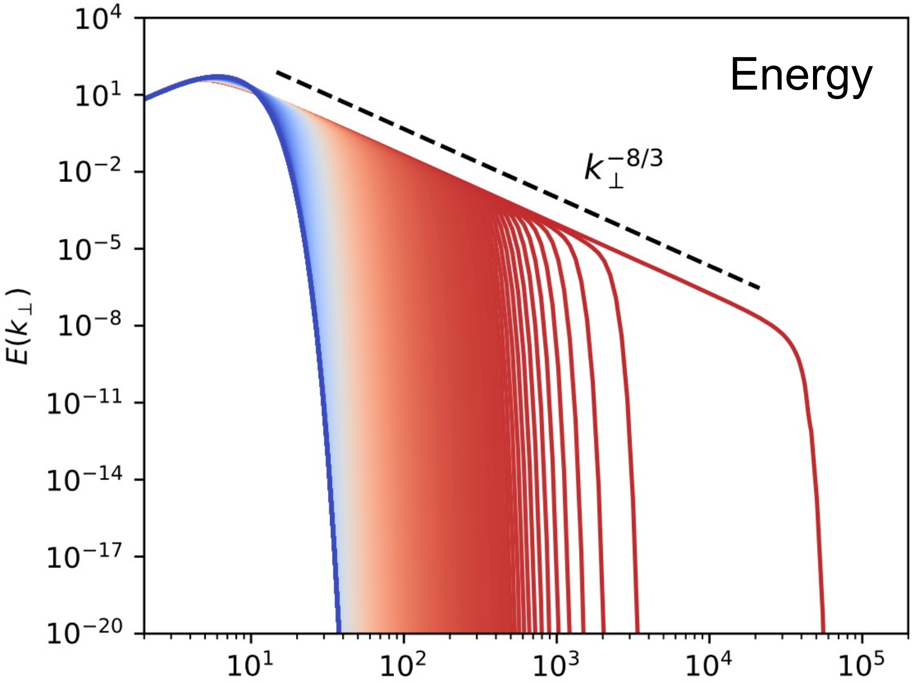
Kinetic Alfvén wave turbulence

Super-local kinetic-Alfvén wave turbulence

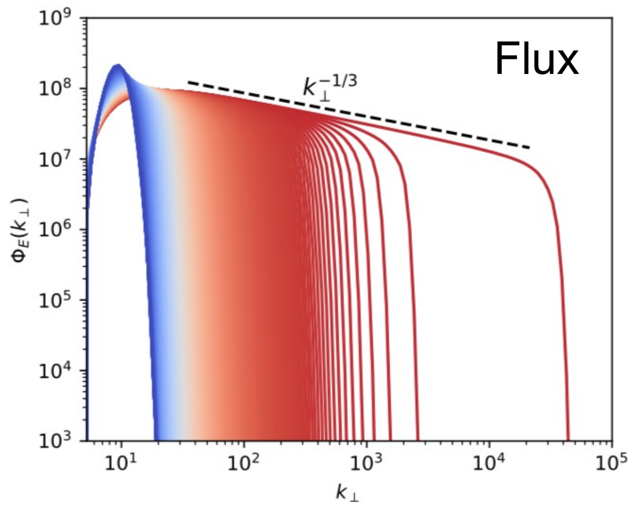
$$\frac{\partial E(k_\perp)}{\partial t} = C\frac{\partial}{\partial k_\perp}\left[k_\perp^7 E(k_\perp)\frac{\partial (E(k_\perp)/k_\perp)}{\partial k_\perp}\right] - \eta k_\perp^6 E(k_\perp)$$

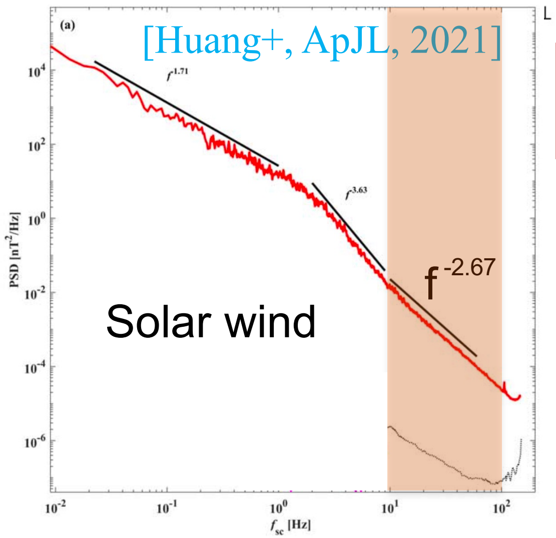
$$\eta = 10^{-16}$$

$$k_{\perp i} = 2^{i/8}$$



[David+, ApJL, 2019]



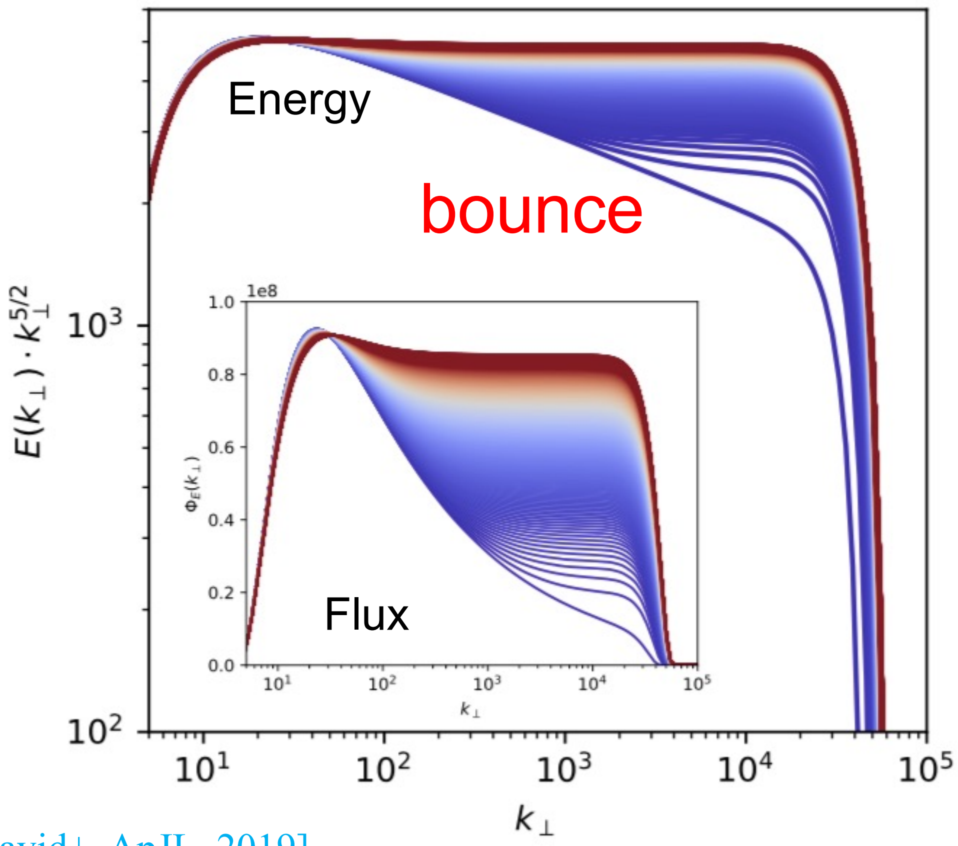


≈ -8/3 emerges as a non-stationary solution !

It is a collisionless solution

Collisionless wave turbulence

The exact stationary solution (-2.5) is reached for t > t_*



[David+, ApJL, 2019]

Viscous effects affect the entire inertial range

but solar wind is free and non-viscous

⇒ ≈ -8/3 can be the solution

Thank you !