Introduction to dynamo theory

Lecture 2

19/02/2024

Plan for this lecture (2 hrs)

1) Observations of magnetic fields in space ✓

2) Modelling magnetized fluids with magnetohydrodynamics ✓

3) Turbulent dynamos - an overview ←

4) Mean-field dynamo

5) Small-scale dynamo

3.2 Dynamos & flows of energy

Governing equations (for an incompressible fluid):

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) - \eta \nabla \times (\nabla \times \vec{B}) \qquad (*)$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = \frac{1}{\rho}\nabla p + \frac{1}{4\pi\rho}(\nabla \times \vec{B}) \times \vec{B} + \frac{1}{m}\vec{F} + \nu \nabla^2 \vec{v} \qquad (**)$$

Multiply $(*)$ by $\frac{\vec{B}}{4\pi}$ and integrate over volume (and ignore fluxes through surfaces):

$$\int \frac{\partial \vec{B}}{\partial t} \cdot \frac{\vec{B}}{4\pi} dV = \int [\nabla \times (\vec{v} \times \vec{B})] \cdot \frac{\vec{B}}{4\pi} dV - \eta \int [\nabla \times (\nabla \times \vec{B})] \cdot \frac{\vec{B}}{4\pi} dV$$

$(\nabla \times \vec{a}) \cdot \vec{b} = \nabla \cdot (\vec{a} \times \vec{b}) + a \cdot (\nabla \times \vec{b})$

$$\Rightarrow \frac{d}{dt} \underbrace{\int \left(\frac{\vec{B}^2}{8\pi}\right) dV}_{\text{magnetic energy}} = \int \frac{1}{4\pi} (\vec{v} \times \vec{B})(\nabla \times \vec{B}) dV - \frac{\eta}{4\pi} \int (\nabla \times \vec{B})^2 dV$$

$\vec{j} = \frac{4\pi}{c} \nabla \times \vec{B}$

$$= \frac{1}{c} \int (\vec{v} \times \vec{B}) \cdot \vec{j} \, dV - \frac{4\pi\eta}{c^2} \int \vec{j}^2 dV$$

$\vec{a} \cdot (\vec{b} \times \vec{c}) = -\vec{b} \cdot (\vec{a} \times \vec{c})$

$$= \underbrace{-\frac{1}{c} \int \vec{v} \cdot (\vec{j} \times \vec{B}) dV}_{\text{work against Lorentz force}} - \underbrace{\frac{1}{\sigma} \int \vec{j}^2 dV}_{\text{resistive losses}}$$

Multiply (xx) by $\rho\vec{v}$ and integrate over volume (and ignore fluxes through surfaces):

$$\underbrace{\frac{d}{dt}\int \frac{1}{2}\rho\vec{v}^2 dV}_{\text{kinetic energy}} = + \underbrace{\frac{1}{c}\int \vec{v}\cdot(\vec{j}\times\vec{B})\,dV}_{\text{work of Lorentz force}} + \int \frac{\rho}{m}\vec{v}\cdot\vec{f}\,dV - 2\nu\int\rho\,(\vec{\nabla}\times\vec{v})^2 dV$$

$\Rightarrow$ "Definition" of a dynamo:

$$\boxed{\text{kinetic energy}} \xrightarrow[\text{= flow of energy}]{\text{MHD dynamo}} \boxed{\text{magnetic energy}}$$

3.3 Different types of dynamos

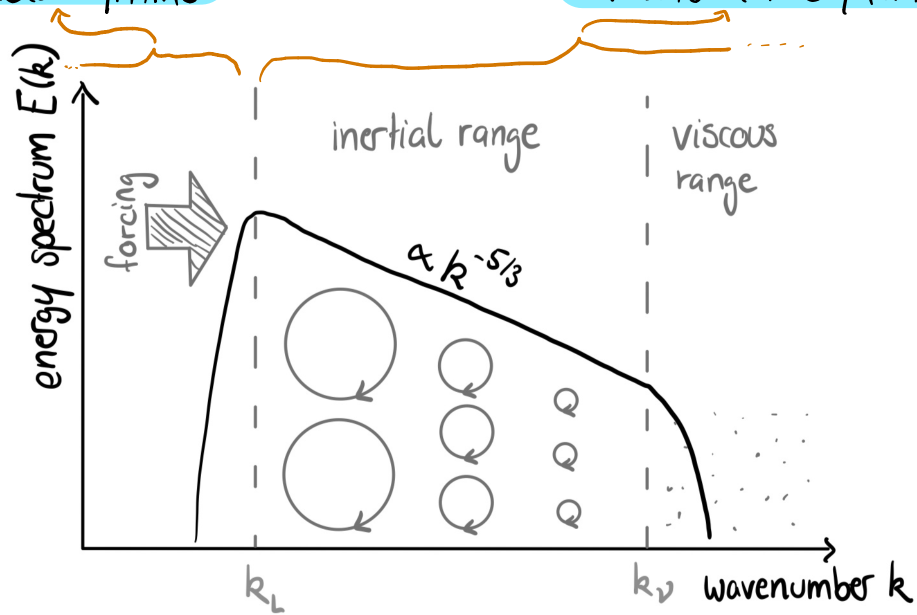
- Kinematic dynamo $\longleftrightarrow$ Nonlinear dynamo

  $\vec{v}$ is given
  $\Rightarrow$ need only (*)

  $\vec{v}$ is affected by $\vec{B}$ [via $\frac{1}{4\pi\rho}(\nabla\times\vec{B})\times\vec{B}$]
  $\Rightarrow$ need (*) and (**)

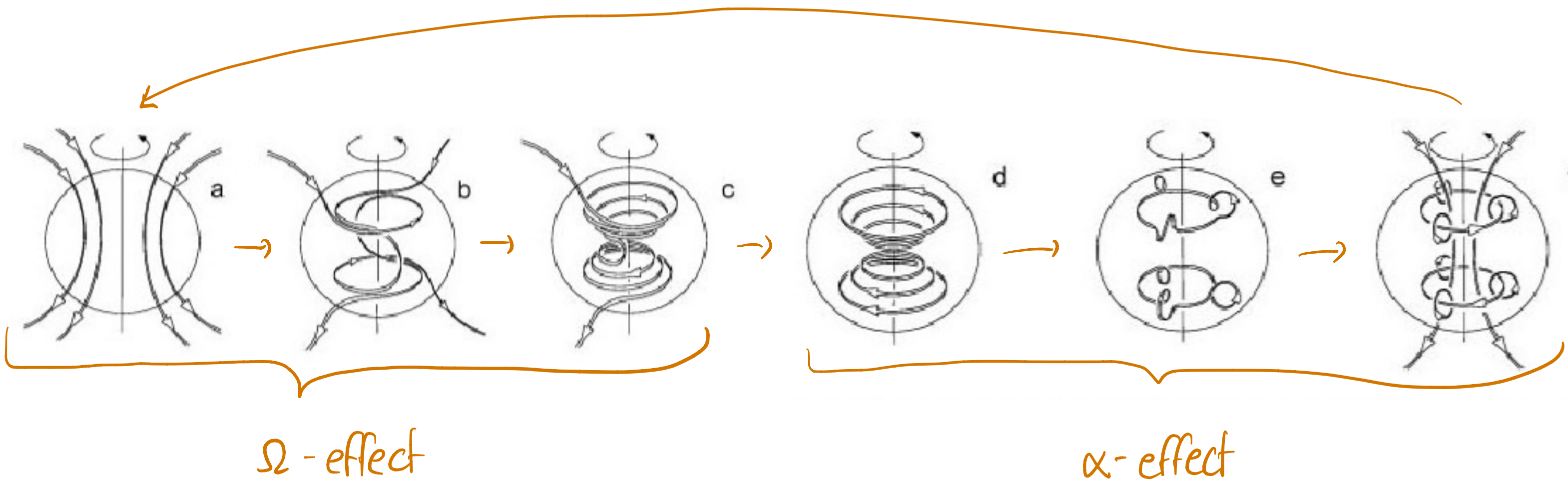
- Large-scale dynamo $\longleftrightarrow$ Small-scale dynamo

  $\hat{=}$ Mean-field dynamo $\qquad$ $\hat{=}$ Fluctuation dynamo



4. Mean-field dynamos

4.1 Sketch



4.2 Mean-field magnetohydrodynamics

→ Steenbeck, Krause & Rädler (1966)

Separation into mean fields and fluctuations: $\vec{B} \to \langle \vec{B} \rangle + \vec{B}'$

$\vec{v} \to \langle \vec{v} \rangle + \vec{v}'$

$\left[\langle x' \rangle = 0; \; \langle c \cdot x \rangle = c \langle x \rangle; \; \langle \langle x \rangle y \rangle = \langle x \rangle \langle y \rangle \right.$

$\left. \langle x + y \rangle = \langle x \rangle + \langle y \rangle; \; \left\langle \frac{\partial x}{\partial t} \right\rangle = \frac{\partial \langle x \rangle}{\partial t} \right]$

Insert in induction equation:

$$\frac{\partial \langle \vec{B} \rangle}{\partial t} + \frac{\partial \vec{B}'}{\partial t} = \nabla \times \left[(\langle \vec{v} \rangle + \vec{v}') \times (\langle \vec{B} \rangle + \vec{B}') \right] + \eta \nabla^2 (\langle \vec{B} \rangle + \vec{B}') \quad (***) \quad | \text{average}$$

$$\Rightarrow \frac{\partial \langle \vec{B} \rangle}{\partial t} = \nabla \times \Big[\underbrace{\langle \langle \vec{v} \rangle \times \langle \vec{B} \rangle \rangle}_{\langle \vec{v} \rangle \times \langle \vec{B} \rangle} + \underbrace{\langle \langle \vec{v} \rangle \times \vec{B}' \rangle}_{0} + \underbrace{\langle \vec{v}' \times \langle \vec{B} \rangle \rangle}_{0} + \underbrace{\langle \vec{v}' \times \vec{B}' \rangle}_{\neq 0} \Big] + \eta \nabla^2 \langle \vec{B} \rangle$$

"mean electromotive force (EMF)" $\equiv \mathcal{E}$

$$\Rightarrow \boxed{\frac{\partial \langle \vec{B} \rangle}{\partial t} = \nabla \times (\langle \vec{v} \rangle \times \langle \vec{B} \rangle) + \nabla \times \mathcal{E} + \eta \nabla^2 \langle \vec{B} \rangle} \quad (****)$$

What is $\mathcal{E}$?

If $\vec{v}'$ and $\vec{B}'$ are uncorrelated: $\mathcal{E} = \langle \vec{v}' \times \vec{B}' \rangle = \langle \vec{v}' \rangle \times \langle \vec{B}' \rangle = 0$

$\Rightarrow$ Is there some correlation? Yes, because $\vec{B}'$ is caused by $\vec{v}'$.

i) Find equation for $\vec{B}'$

Subtract (****) from (***):

$$\frac{\partial \vec{B}'}{\partial t} = \nabla \times \left[\langle \vec{v} \rangle \times \vec{B}' + \vec{v}' \times \langle \vec{B} \rangle + \vec{v}' \times \vec{B}' - \mathcal{E} \right] + \eta \nabla^2 \vec{B}'$$

ii) First-order smoothing approximation: $\vec{B}'$ remains small during correlation time τ

$\Rightarrow$ ignore all terms linear in $\vec{B}'$ (incl. $\mathcal{E}$)

$$\Rightarrow \frac{\vec{B}'}{\tau} \approx \nabla \times \left(\vec{v}' \times \langle \vec{B} \rangle \right)$$

$$\Leftrightarrow \vec{B}' \approx \tau \left(\langle \vec{B} \rangle \cdot \nabla \right) \vec{v}' - \tau \left(\vec{v}' \cdot \nabla \right) \langle \vec{B} \rangle$$

Use to evaluate EMF:

$$\mathcal{E}_i = \langle \vec{v}\,' \times \vec{B}\,' \rangle_i$$

$$= \langle \varepsilon_{ijk} v_j' B_k' \rangle$$

| insert $\vec{B}\,'$

$$= \left\langle \varepsilon_{ijk} v_j' \tau \langle B \rangle_l \frac{\partial v_k'}{\partial x_l} \right\rangle - \left\langle \varepsilon_{ijk} \tau v_j' v_l' \frac{\partial \langle B \rangle_k}{\partial x_l} \right\rangle$$

$$= \underbrace{\varepsilon_{ijk} \left\langle v_j' \frac{\partial v_k'}{\partial x_l} \right\rangle \tau}_{\equiv \alpha_{il}} \langle B \rangle_l - \underbrace{\varepsilon_{ijk} \langle v_j' v_l' \rangle \tau}_{\equiv -\beta_{ikl}} \frac{\partial \langle B \rangle_k}{\partial x_l}$$

depend only on statistical properties of velocity field

iii) Assume isotropic turbulence

$$\alpha_{il} = \alpha\,\delta_{il} \quad \Rightarrow \quad \boxed{\alpha = -\frac{1}{3}\langle \vec{v}' \cdot (\nabla \times \vec{v}') \rangle \tau}$$

$$\beta_{ikl} = -\eta_T\,\varepsilon_{ikl} \quad \Rightarrow \quad \boxed{\eta_T = \frac{1}{3}\langle \vec{v}' \cdot \vec{v}' \rangle \tau}$$

$$\Rightarrow \boxed{\frac{\partial \langle \vec{B} \rangle}{\partial t} = \nabla \times (\langle \vec{v} \rangle \times \langle \vec{B} \rangle) + \nabla \times (\alpha \langle \vec{B} \rangle) + (\eta + \eta_T) \nabla^2 \langle \vec{B} \rangle}$$

Simple dynamo solutions

- α^2 dynamo : $\langle \vec{v} \rangle = 0$

$$\Rightarrow \frac{\partial \langle \vec{B} \rangle}{\partial t} = \nabla \times (\alpha \langle \vec{B} \rangle) + (\eta + \eta_T) \nabla^2 \langle \vec{B} \rangle$$

Ansatz : $\langle \vec{B} \rangle (x) = \hat{\vec{B}}(\vec{k}) \exp(i\vec{k}\vec{x} + \gamma t)$

$$\Rightarrow \gamma \hat{\vec{B}} = \alpha i \vec{k} \times \hat{\vec{B}} - (\eta + \eta_T) k^2 \hat{\vec{B}}$$

$$\Rightarrow \gamma \hat{\vec{B}} = \begin{pmatrix} -(\eta+\eta_T)k^2 & -i\alpha k_z & i\alpha k_y \\ i\alpha k_z & -(\eta+\eta_T)k^2 & -i\alpha k_x \\ -i\alpha k_y & i\alpha k_x & -(\eta+\eta_T)k^2 \end{pmatrix} \hat{\vec{B}}$$

$$\Rightarrow \left(\gamma + (\eta+\eta_T)k^2\right)\left[\left(\gamma + (\eta+\eta_T)k^2\right)^2 - \alpha^2 k^2\right] = 0$$

$$\Rightarrow \gamma_0 = -(\eta+\eta_T)k^2 \quad \& \quad \boxed{\gamma_\pm = \pm |\alpha k| - (\eta+\eta_T)k^2}$$

$$\frac{d\gamma_+}{dk} = -2(\eta+\eta_T)k + |\alpha| \overset{!}{=} 0$$

$$\Rightarrow \boxed{k_{max} = \frac{|\alpha|}{2(\eta+\eta_T)}}$$

$$\gamma_{max} = \left|\frac{\alpha^2}{2(\eta+\eta_T)}\right| - \frac{\alpha^2}{4(\eta+\eta_T)}$$

$$\Rightarrow \boxed{\gamma_{max} = \frac{\alpha^2}{4(\eta+\eta_T)}}$$

- $\alpha\Omega$-dynamo: $\quad \langle \vec{v} \rangle = (0, S \cdot x, 0)$

shear, e.g. $S = -\frac{3}{2}\Omega$ for Keplerian disc

$S = r \frac{\partial \Omega}{\partial r}$ in Sun

Growth rates:

$\Rightarrow \gamma_0 = -(\eta + \eta_T) k^2$

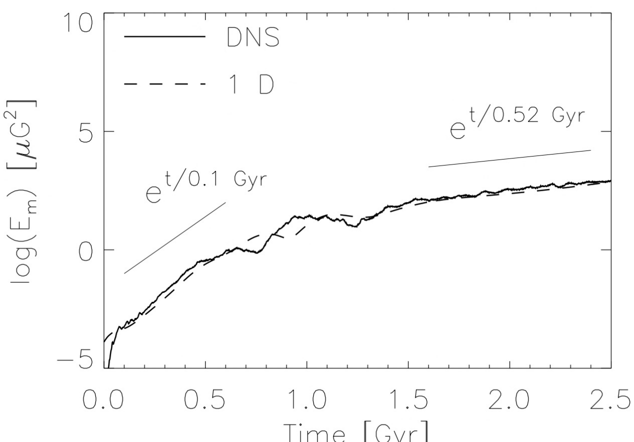
$$\boxed{\gamma_{\pm} = \pm \left| \frac{1}{2} \alpha S k_z \right|^{1/2} - (\eta + \eta_T) k^2}$$

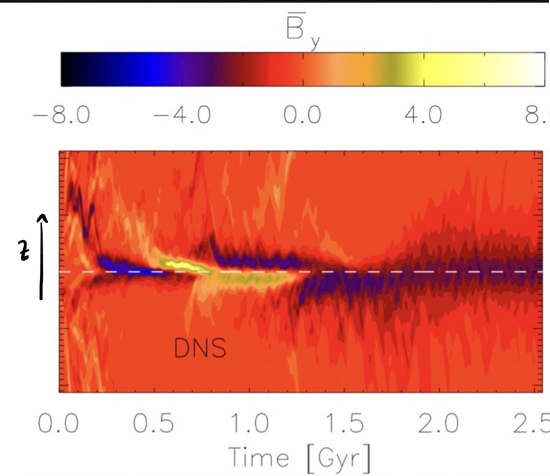
for axisymmetric solutions $k_y = 0$

Oscillations:

$$\boxed{\omega_{\pm} = \left| \frac{1}{2} \alpha S k_z \right|^{1/2}}$$

4.3 Astrophysical dynamo simulations



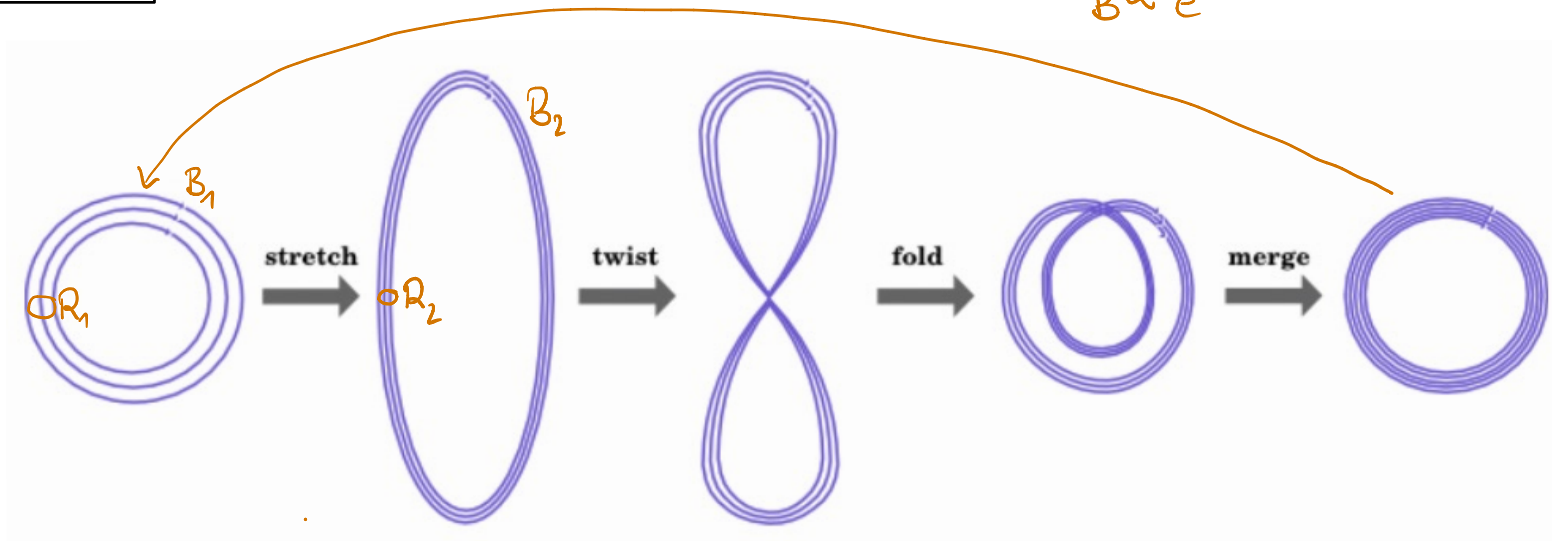


[Bendre et al. 2020]

5. Small-scale dynamo

5.1 Sketch

$B \propto e^{\Gamma t}$

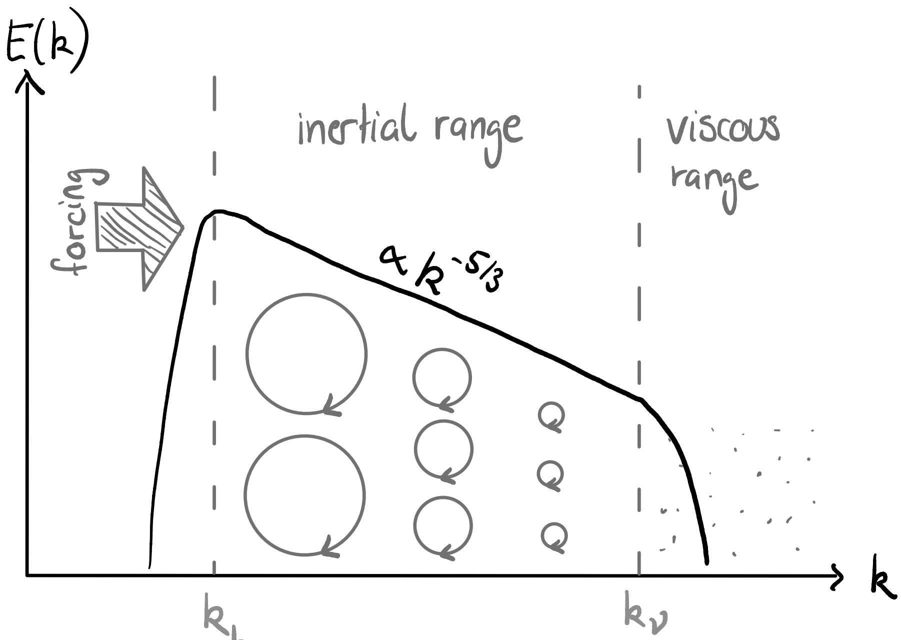


$B_1 R_1^2 = B_2 R_2^2$

$\Rightarrow B_2 > B_1$

5.2 Phenomenology

Growth rate of the small-scale dynamo



$\approx Re^{3/4} k_L \Rightarrow l_\nu = Re^{-3/4} L$

Growth rate = inverse of turbulent eddy time scale

$$\Gamma \approx \frac{v_\nu}{l_\nu}$$

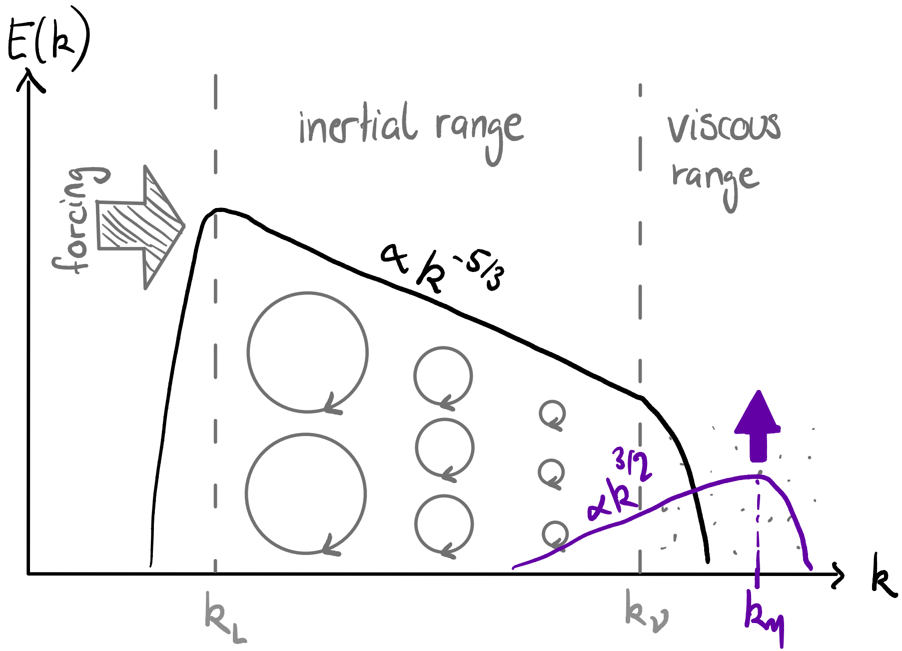
$| v(l) \propto l^{1/3}$

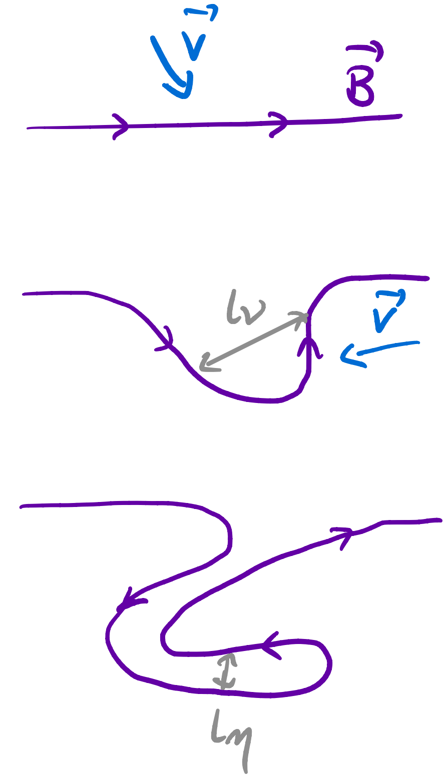
$$= \frac{V}{L^{1/3}} \cdot \frac{l_\nu^{1/3}}{l_\nu}$$

$$= \frac{V}{L} \cdot \left(\frac{l_\nu}{L}\right)^{-2/3}$$

$$\Rightarrow \boxed{\Gamma = \frac{V}{L} \cdot Re^{1/2}}$$

Buildup of magnetic energy



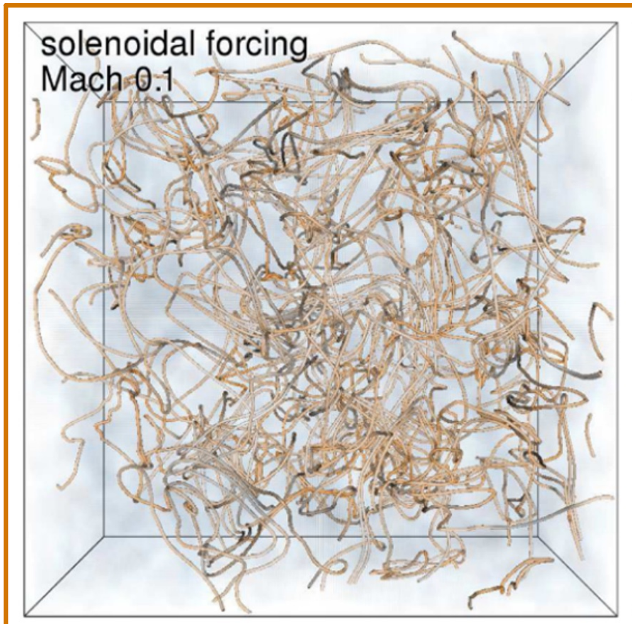


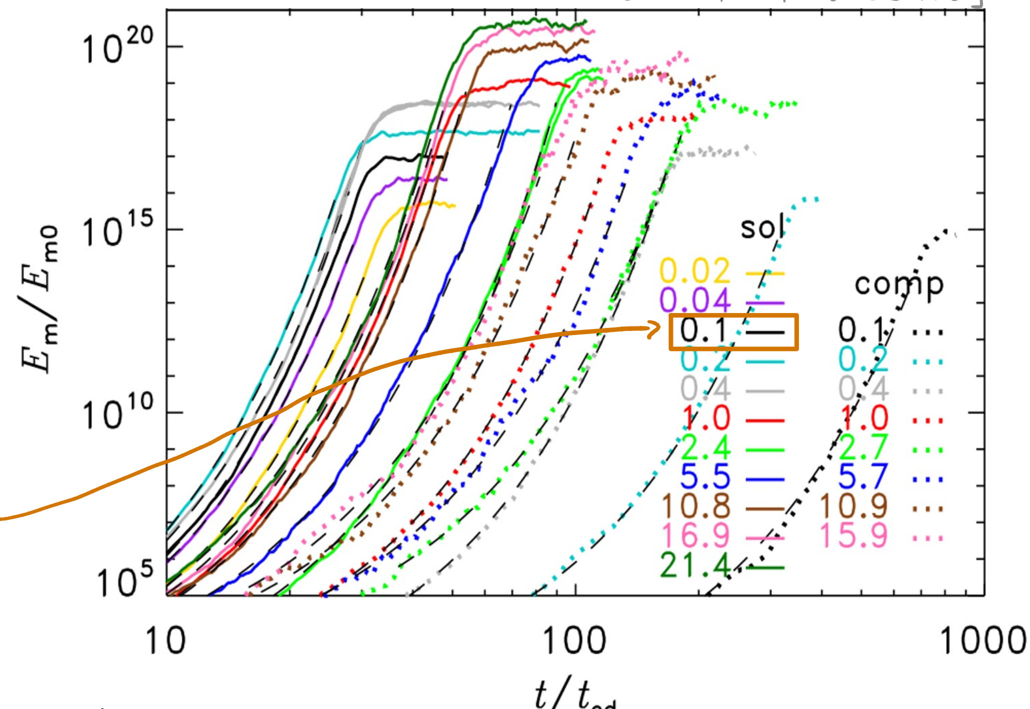
Detailed theory: Kazantsev 1968

5.3 Small-scale dynamo simulations

Dynamo in periodic box with driven turbulence

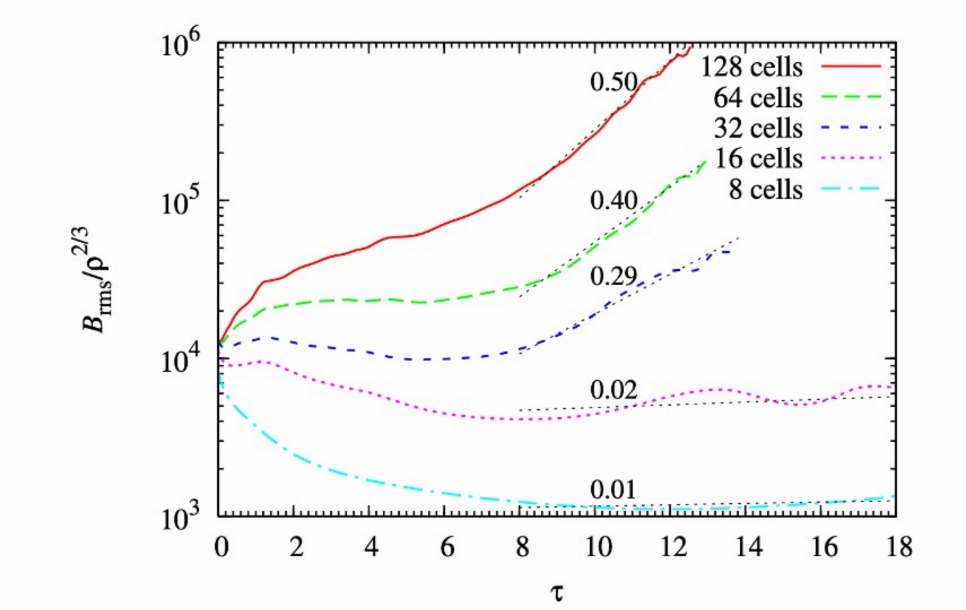
[Federrath et al. 2011b]





⇒ Dynamo growth rate and saturation depend on plasma parameters.

Dynamo in turbulence from gravitational collapse



[Federrath et al. 2011a]

$\Rightarrow$ Minimum resolution ($\hat{=}$ minimum Re_M) needed in simulations to see the small-scale dynamo.

SUMMARY

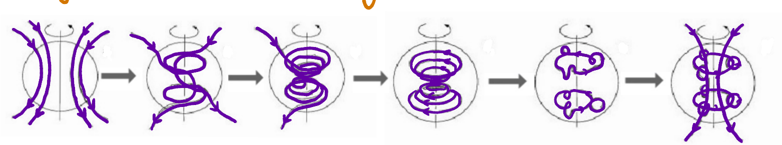
1. The Universe is filled with magnetic fields.

2. The best known candidate to explain the amplification of weak seed magnetic fields to the observed field strengths are MHD dynamos.

3. Dynamos convert (turbulent) kinetic energy to magnetic energy exponentially.

4. We distinguish large-scale (≙ mean-field) dynamos and small-scale dynamos.

generate ordered magnetic fields slowly



generate random magnetic fields quickly