Angular momentum transport in viscous accretion discs

Giuseppe Lodato - Les Houches Winter School on the Physics of Star Formation - 14-16 February 2024

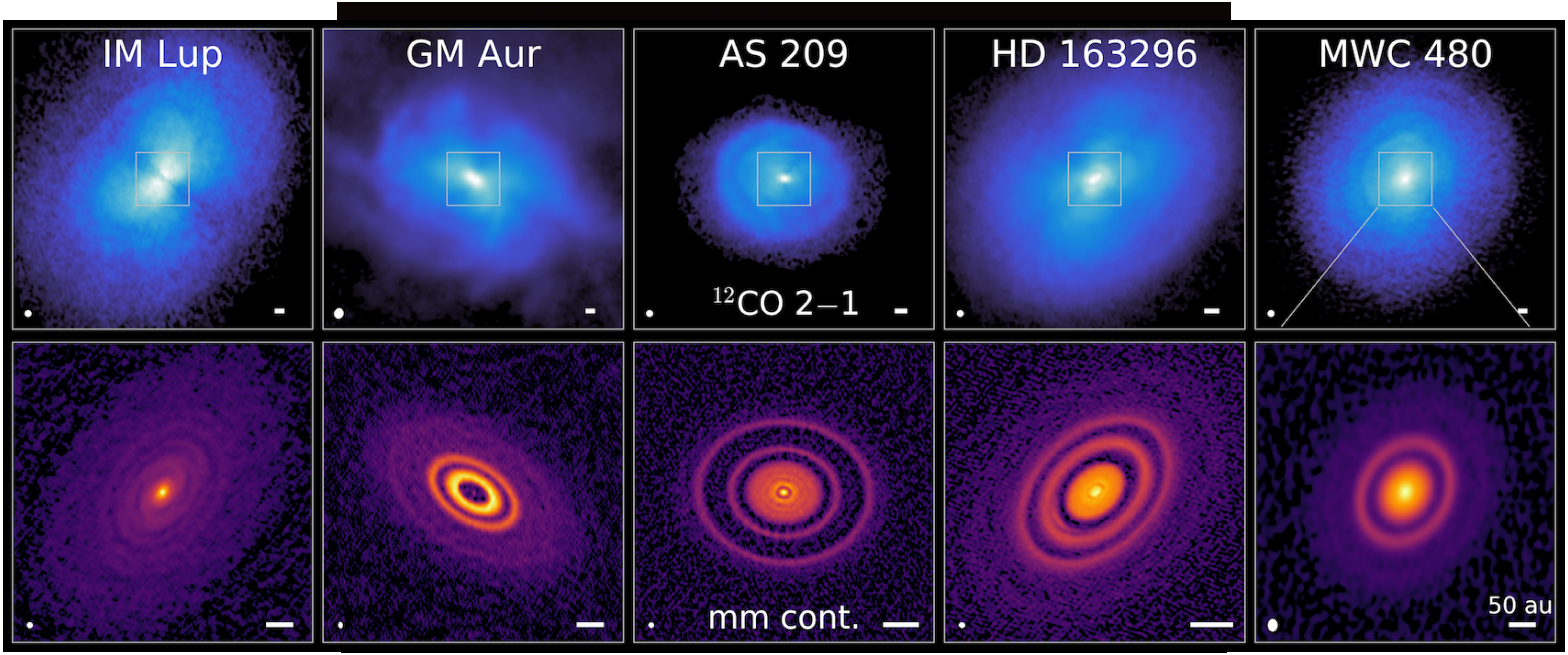
Outline

- Lecture 1: Fundamentals of disc accretion
- Lecture 2: Basic equations and simple analytical solutions
- Lecture 3: Viscosity and turbulent transport
- Lecture 4: The magneto-rotational and the gravitational instability in discs

Accretion discs in Astrophysics

- Accretion discs play a fundamental role in several different contexts in astrophysics
  - central engine in AGN
  - compact objects in galactic binary systems (X-ray binaries, Dwarf novae, etc.)
  - Protostellar discs

Varous kinds of accretion discs



Gas and dust emission from protostellar accretion discs from ALMA

Exercises for lecture 1

1. Demonstrate that a <u>radially and vertically</u> isothermal disc rotates on cylinders (Poincarè-Wavre theorem)

- Hints:
  - Demonstrate that the pressure term at z>0 gives a super-Keplerian rotation term
  - Compute the pressure correction based on the usual Gaussian profile
  - Compute the full azimuthal velocity

$$v_\phi^2 = v_\mathrm{K}^2 \left[1 - \gamma' \left(\frac{H}{R}\right)^2 - q\left(1 - \frac{1}{\sqrt{1 + z^2/R^2}}\right)\right]$$

2. Compute the effect of an exponential truncation to the disc on rotation velocity

Summary - Lecture 1

- Basic disc equations:

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}\left(R\Sigma v_R\right) = 0$$

Continuity equation

$$\frac{\partial}{\partial t}\left(\Sigma R v_\phi\right) + \frac{1}{R}\frac{\partial}{\partial R}\left(R v_r \Sigma R v_\phi\right) = \frac{1}{R}\frac{\partial}{\partial R}\left(\nu \Sigma R^3 \Omega'\right)$$

Angular momentum conservation

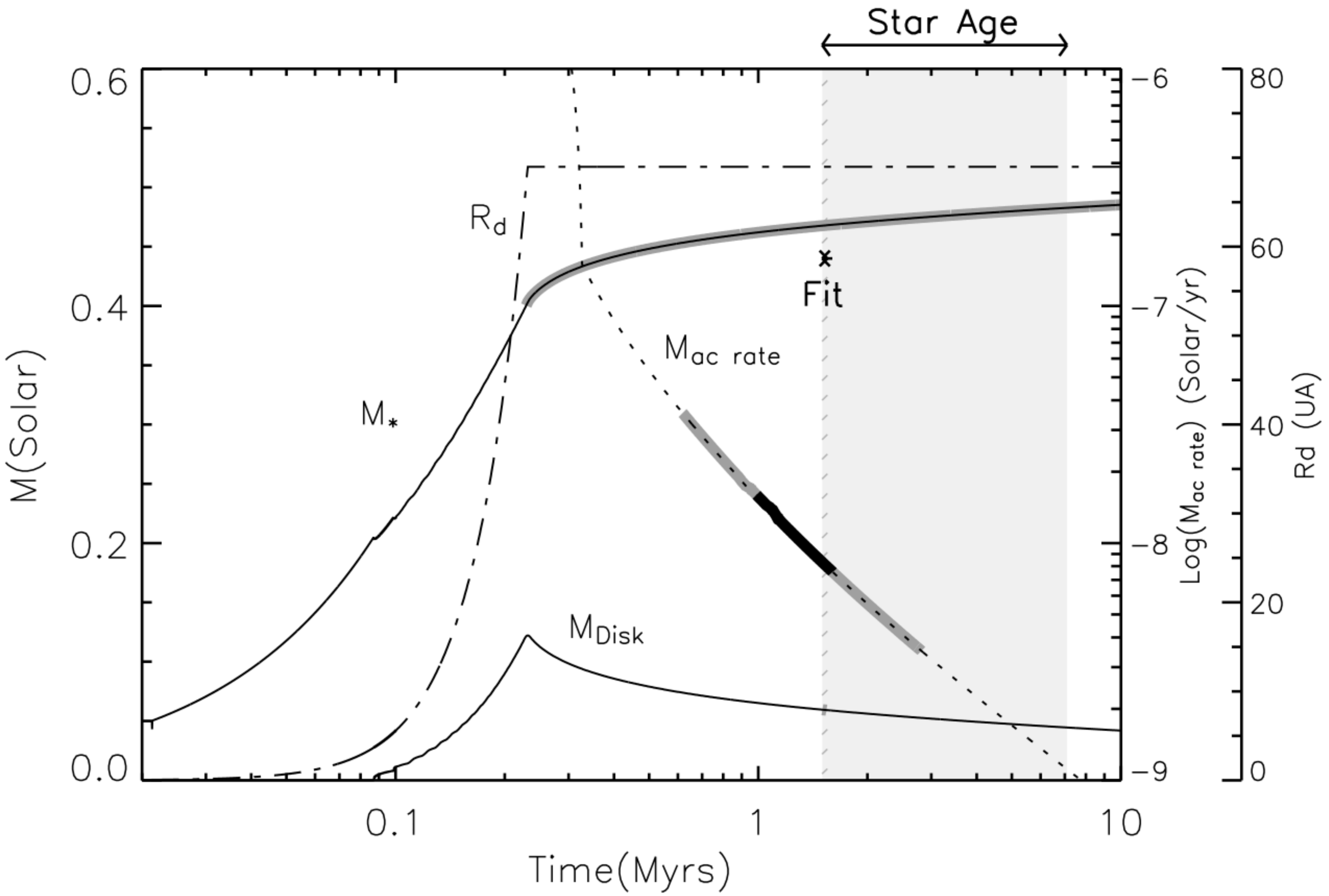
$$\frac{H}{R} = \frac{c_\mathrm{s}}{v_\mathrm{K}}$$

Vertical hydrostatic balance

$$v_\phi^2 = v_\mathrm{K}^2\left[1 - \beta\left(\frac{H}{R}\right)^2\right]$$

Radial centrifugal balance

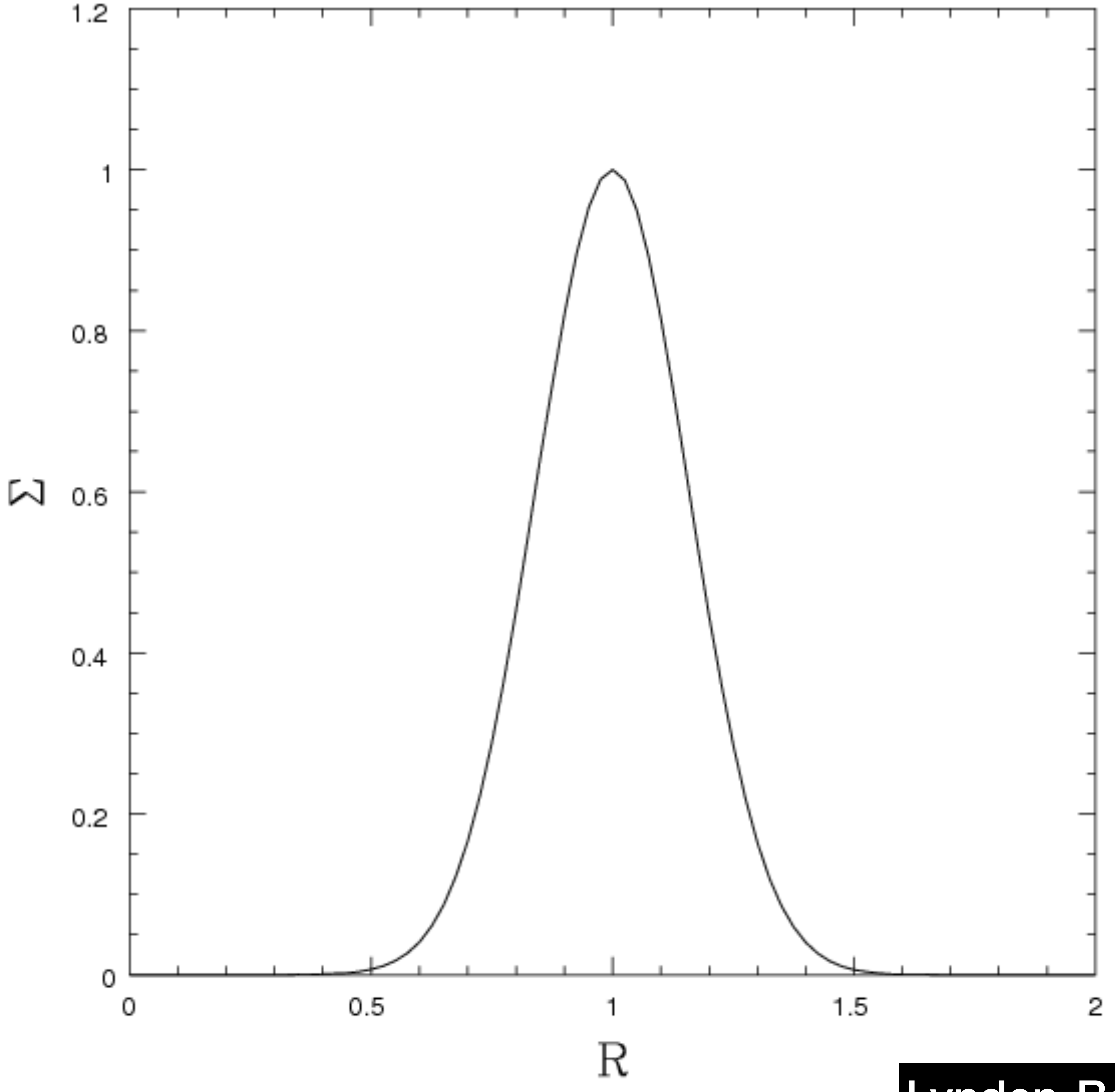
Self-gravitating discs?



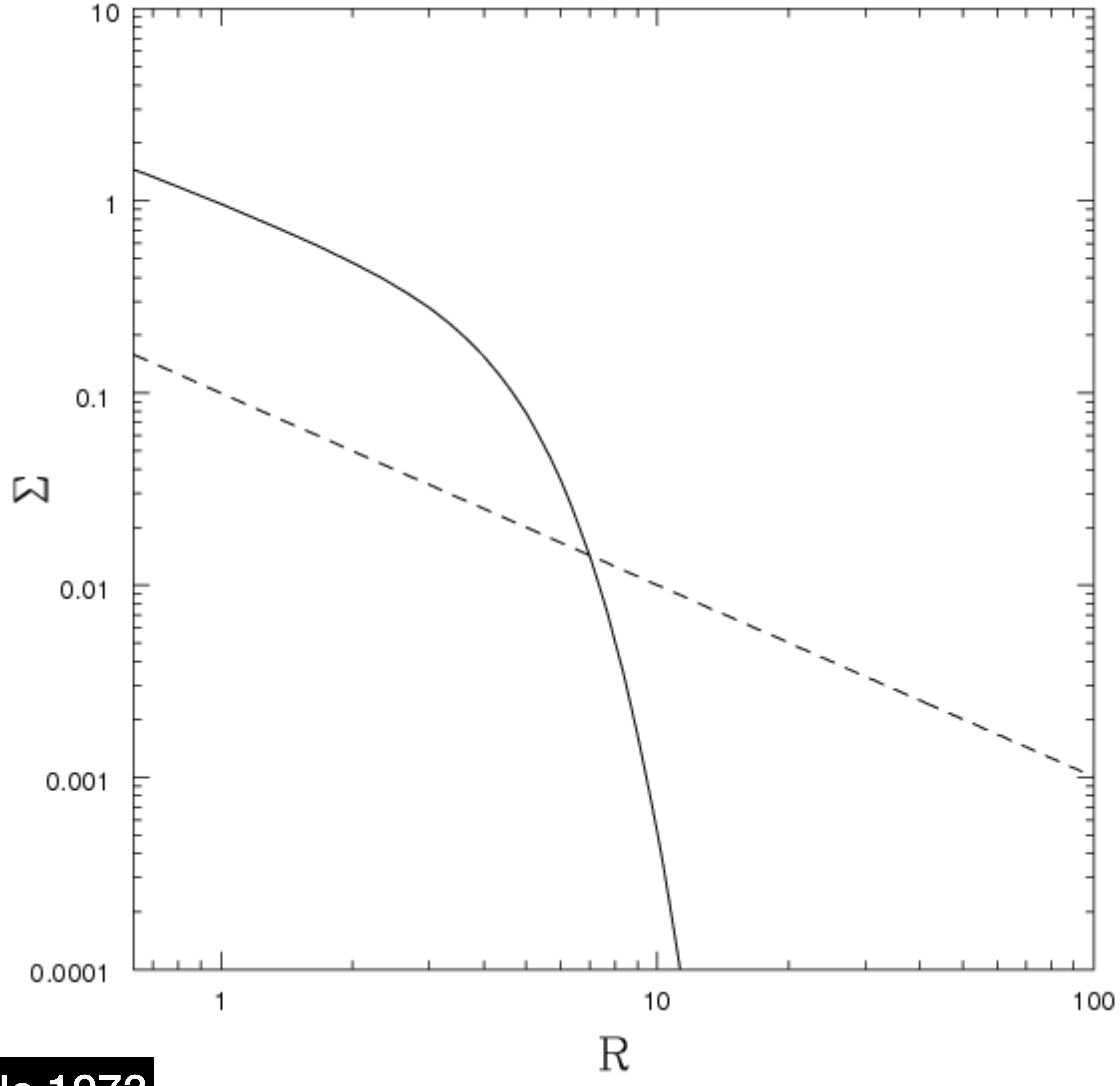
Hueso & Guillot 2005

Typical viscous evolution

The “spreading ring”



The self-similar solution



Lynden-Bell & Pringle 1973

Viscous evolution of mass and accretion rate

$$\Sigma(R,t) = \frac{M_0}{2\pi R_\mathrm{c}^2}(2-\gamma)\left(\frac{R}{R_\mathrm{c}}\right)^{-\gamma} T^{-\eta} \exp\left(-\frac{\left(R/R_\mathrm{c}\right)^{(2-\gamma)}}{T}\right)$$

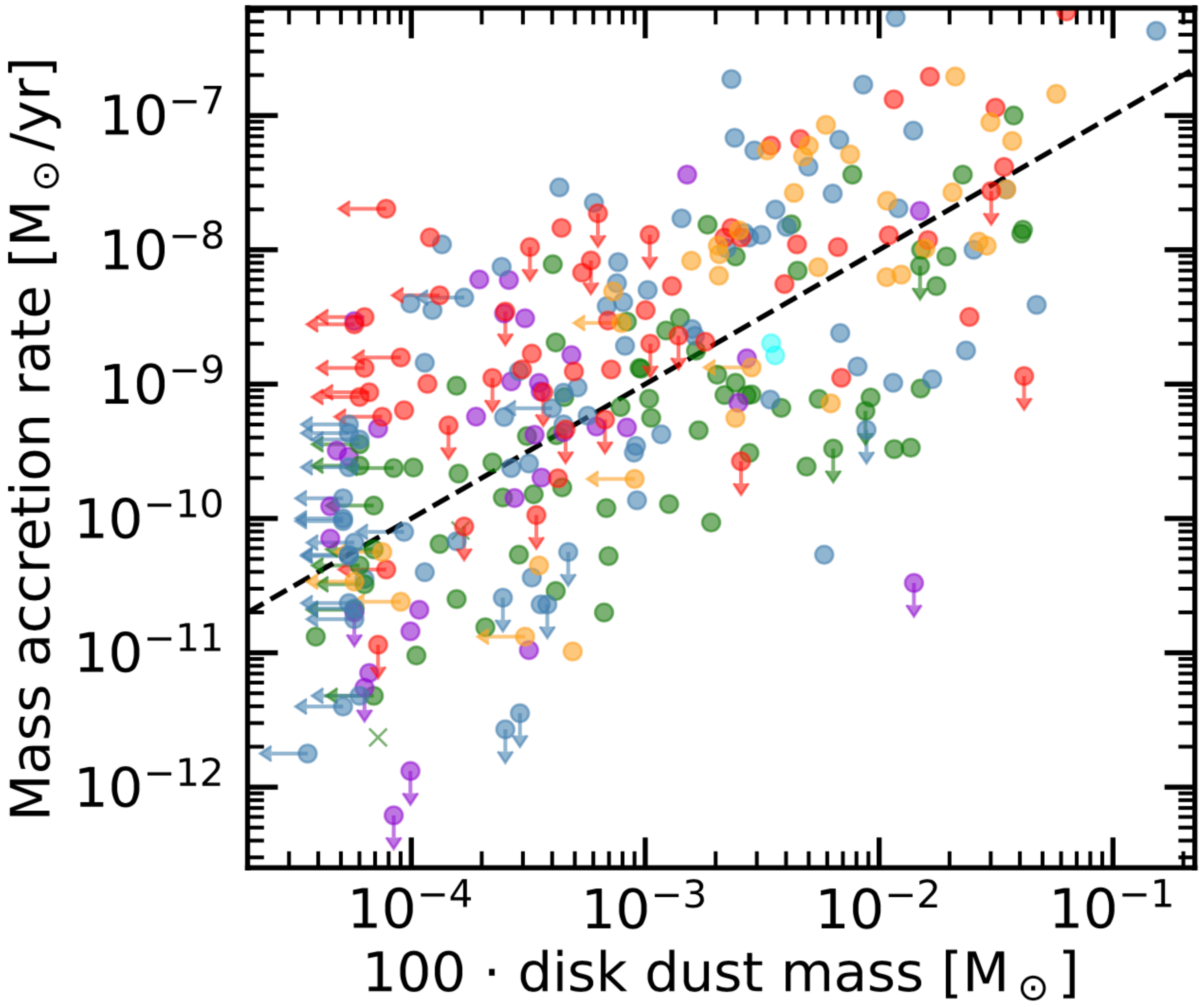
$$M_\mathrm{d}(t) = M_0 T^{(1-\eta)}$$

$$\eta = \frac{5/2-\gamma}{2-\gamma}$$

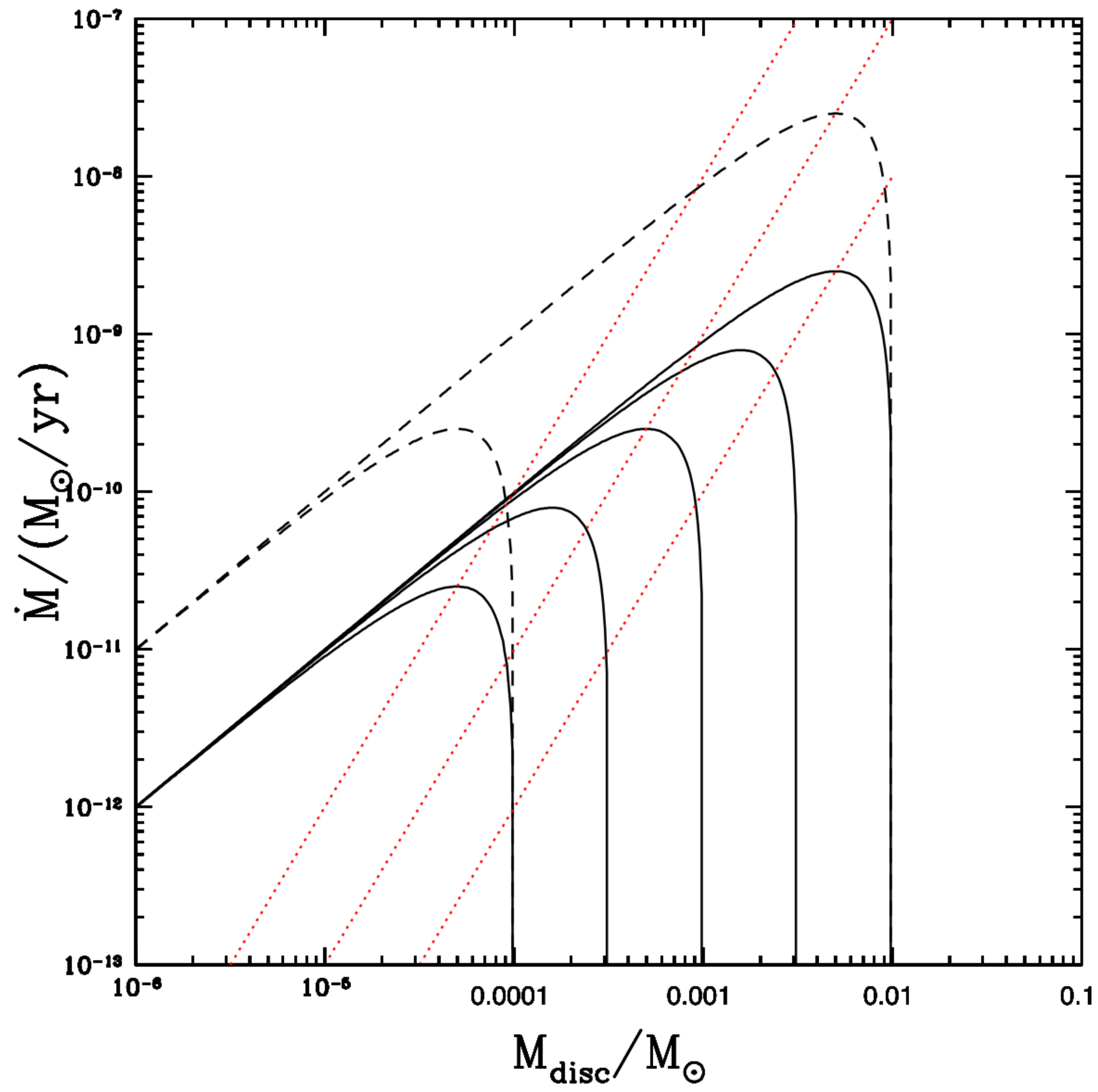
$$\dot{M} = -\frac{\mathrm{d}M_\mathrm{d}}{\mathrm{d}t} = (\eta-1)\frac{M_0}{t_\nu}T^{-\eta}$$

$$t_\mathrm{disc} = \frac{M_\mathrm{d}(t)}{\dot{M}_\mathrm{acc}(t)} = 2(2-\gamma)\left(t+t_\nu\right)$$

Correlation between disc properties



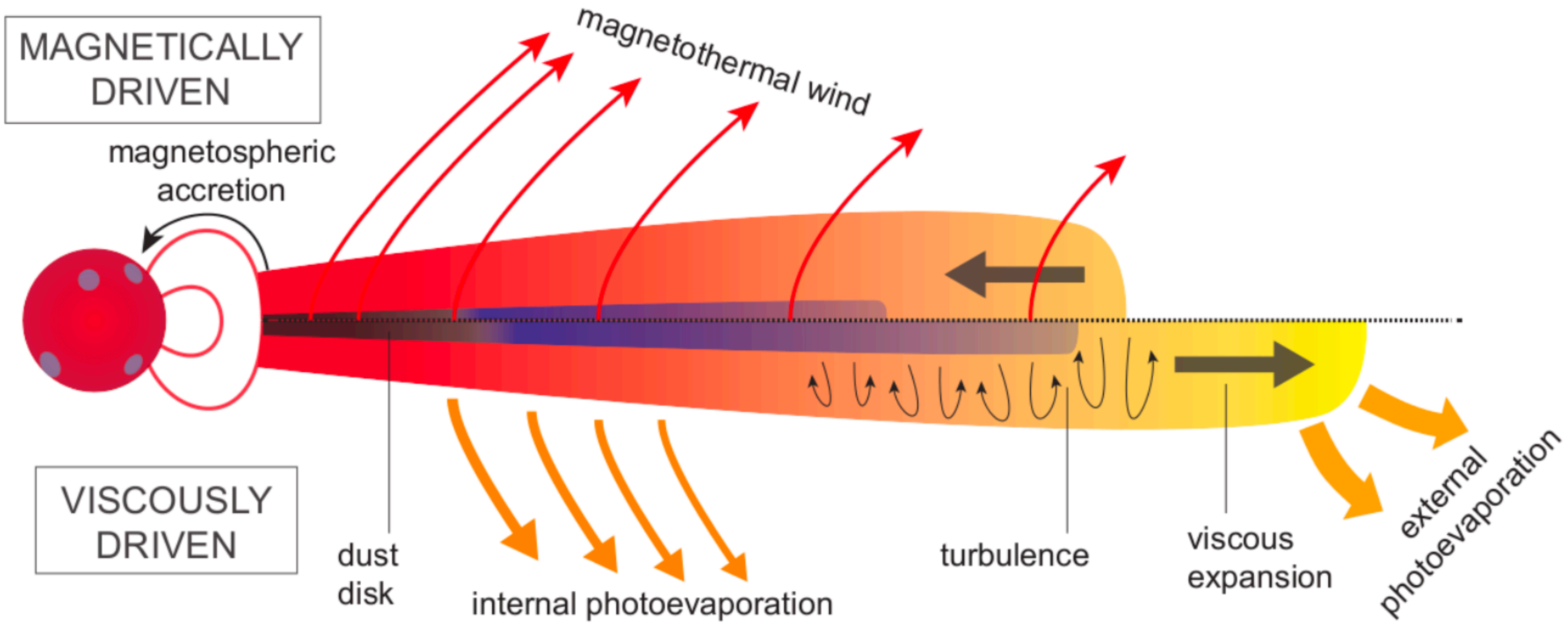
Manara et al PPVII 2023



Lodato et al 2017

$$t_{\rm disc} = \frac{M_{\rm d}(t)}{\dot{M}_{\rm acc}(t)} = 2(2-\gamma)\left(t + t_\nu\right)$$

Various physical effects in disc evolution

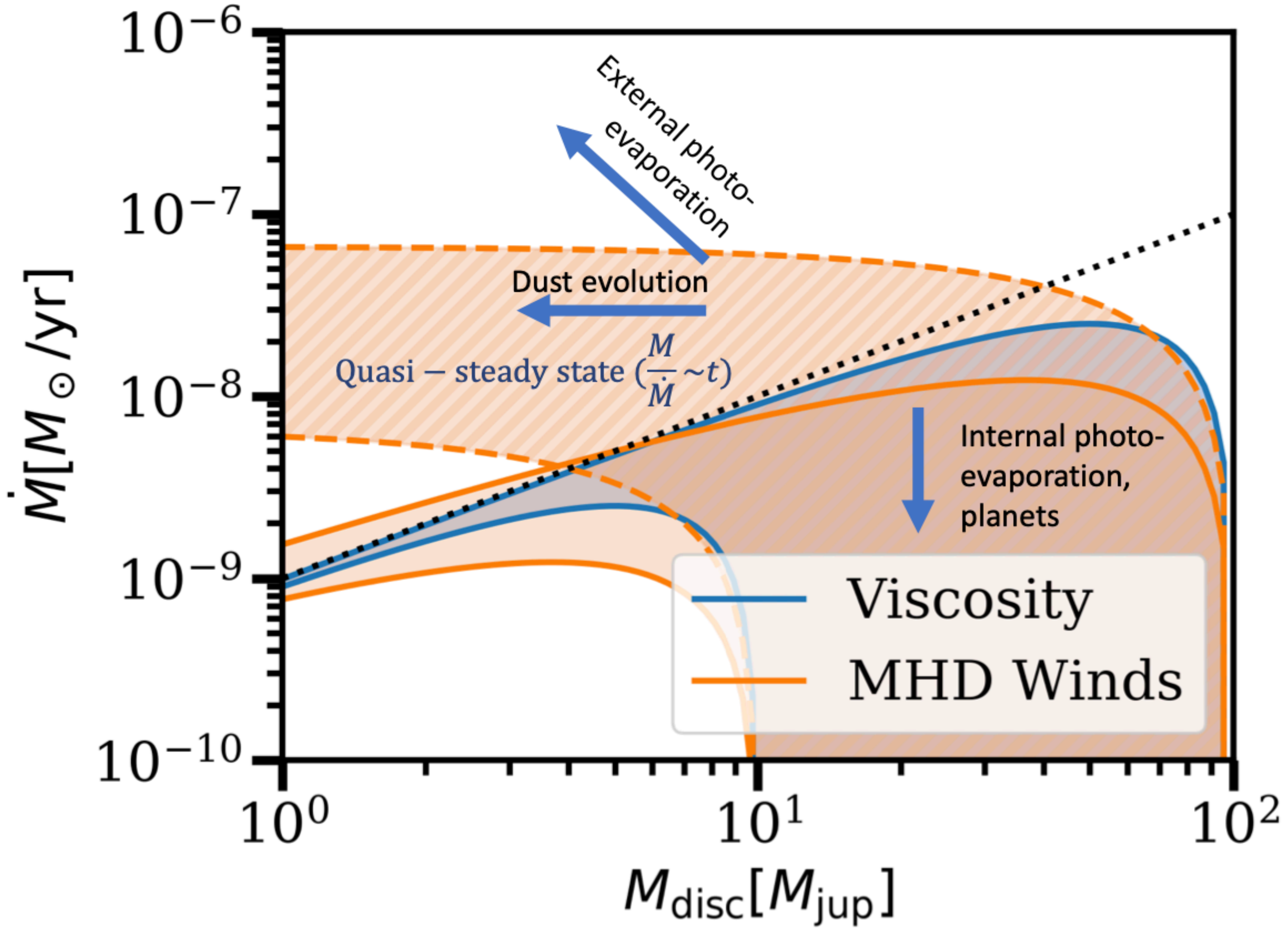


Manara et al PPVII 2023

Evolution due to MHD disc winds

Tabone et al 2022

$$\frac{\partial\Sigma}{\partial t} = \frac{3}{R}\frac{\partial}{\partial R}\left\{\underbrace{\frac{1}{R\Omega}\frac{\partial}{\partial R}\left(R^2\alpha_{\rm SS}\Sigma c_s^2\right)}_{\text{Viscous diffusion}}\right\} + \frac{3}{2R}\frac{\partial}{\partial R}\left\{\underbrace{\frac{\alpha_{\rm DW}\Sigma c_s^2}{\Omega}}_{\text{Advection induced by MHD winds}}\right\} - \underbrace{\frac{3\alpha_{\rm DW}\Sigma c_s^2}{4(\lambda-1)R^2\Omega}}_{\text{Mass removal by MHD winds}}$$



Summary - Lecture 2

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{R}\frac{\partial}{\partial R}\left(R^{1/2}\frac{\partial}{\partial R}\left(\nu \Sigma R^{1/2}\right)\right)$$

Disc evolution due to viscosity - diffusion

$$\Sigma(R,t) = \frac{M_0}{2\pi R_\mathrm{c}^2}(2-\gamma)\left(\frac{R}{R_\mathrm{c}}\right)^{-\gamma} T^{-\eta} \exp\left(-\frac{\left(R/R_\mathrm{c}\right)^{(2-\gamma)}}{T}\right)$$

Self-similar solution (Lynden-Bell & Pringle 1973)

$$D(R) = \nu \Sigma (R\Omega')^2$$

Viscous heating

$$\sigma_\mathrm{SB} T^4 = \frac{3}{8\pi}\frac{GM\dot{M}}{R^3}$$

Temperature profile of a viscously heated disc

Exercise for Lecture 2

- A malevolent deity suddenly removes the outer parts of a disc, reducing its outer radius from R_1 to $R_2<R_1$
- What happens to the disc lifetime?

A. The viscous timescale at R_2 is smaller than the one at R_1: the lifetime is reduced

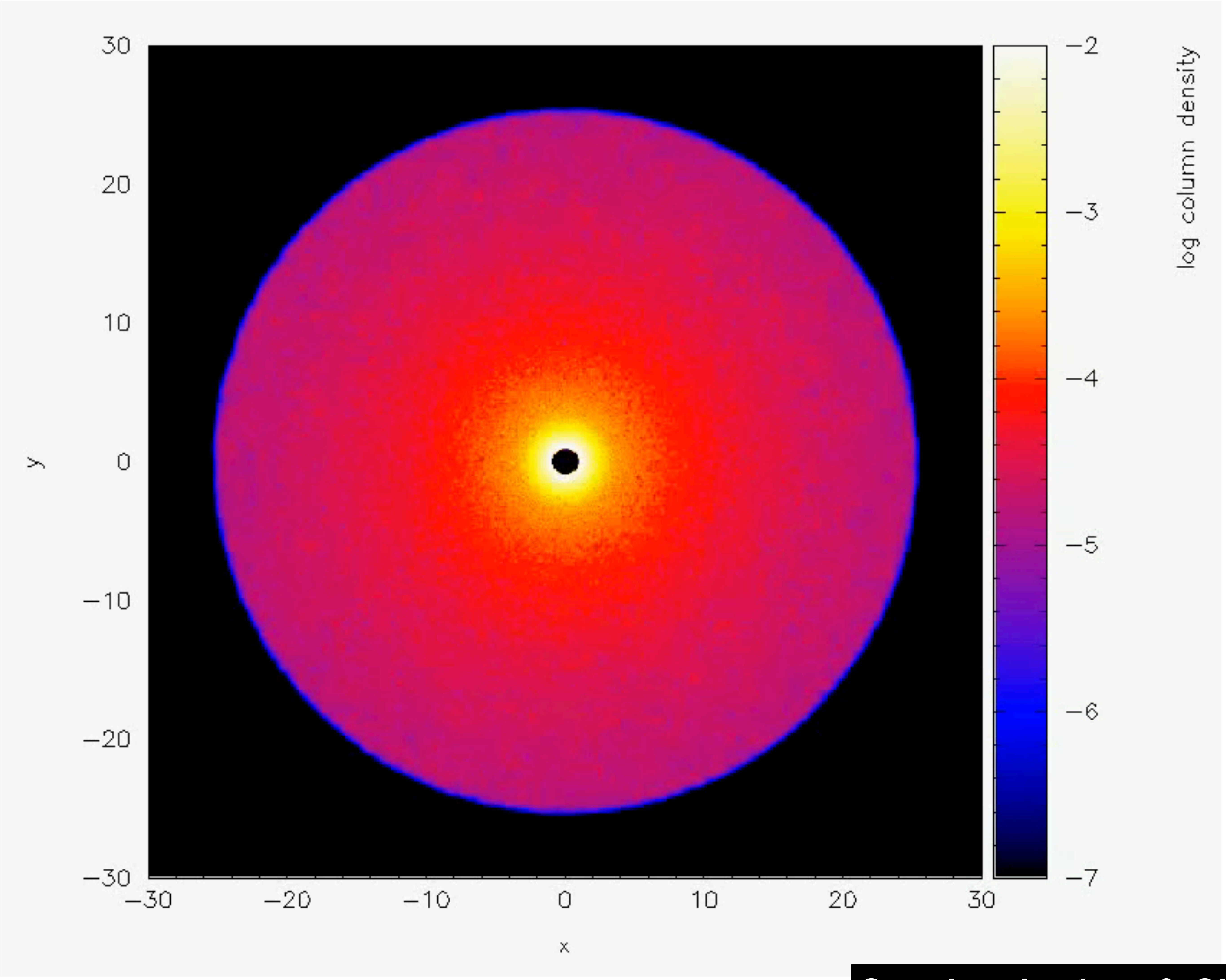
B. The viscous timescale is a local property and does not depend on whether there is or not an outer disc: the lifetime stays the same

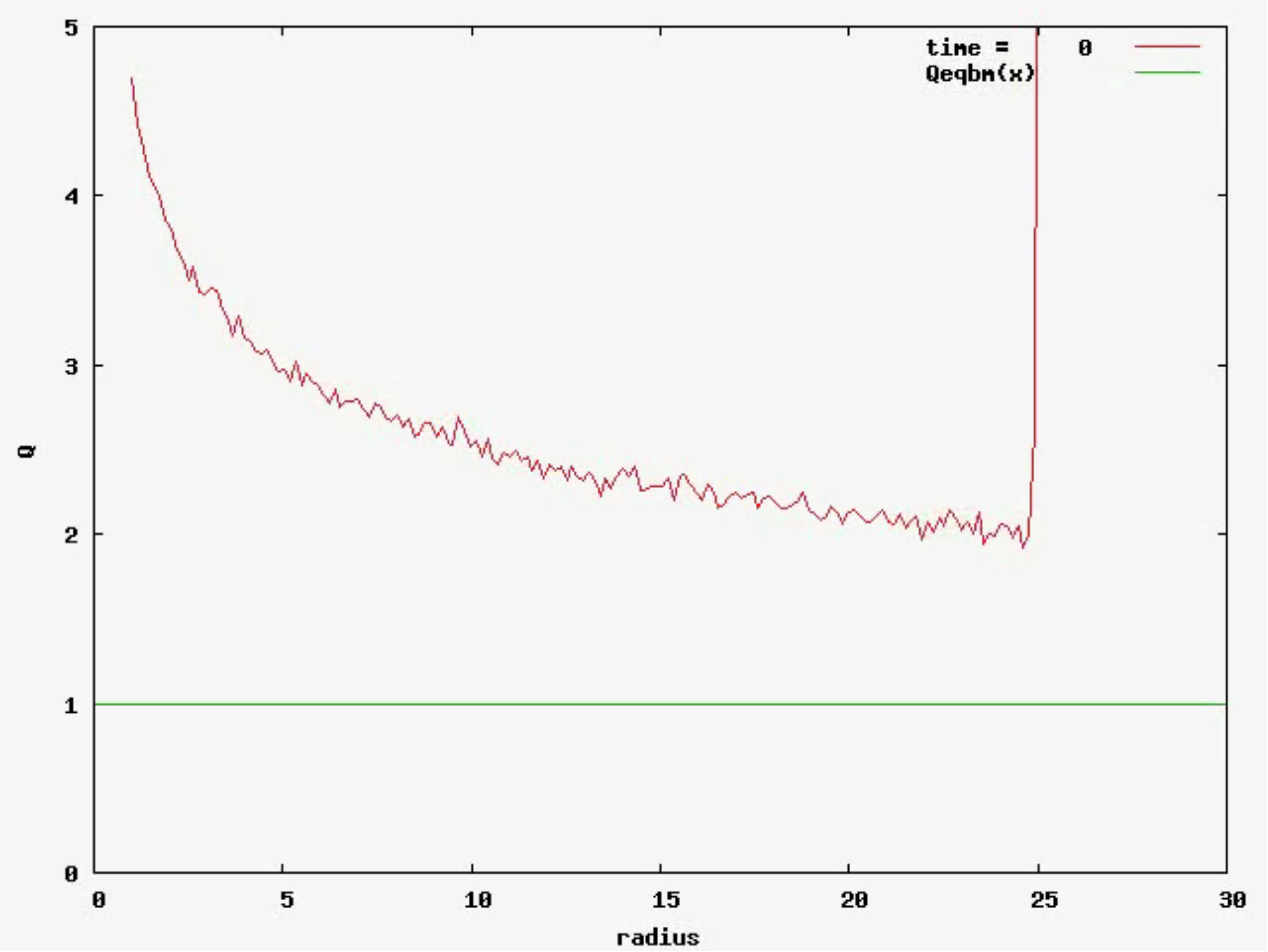
C. After truncation, the disc expands beyond R_2 where the viscous timescale is longer: the lifetime is increased

The magneto-rotational instability

Credit Phil Armitage

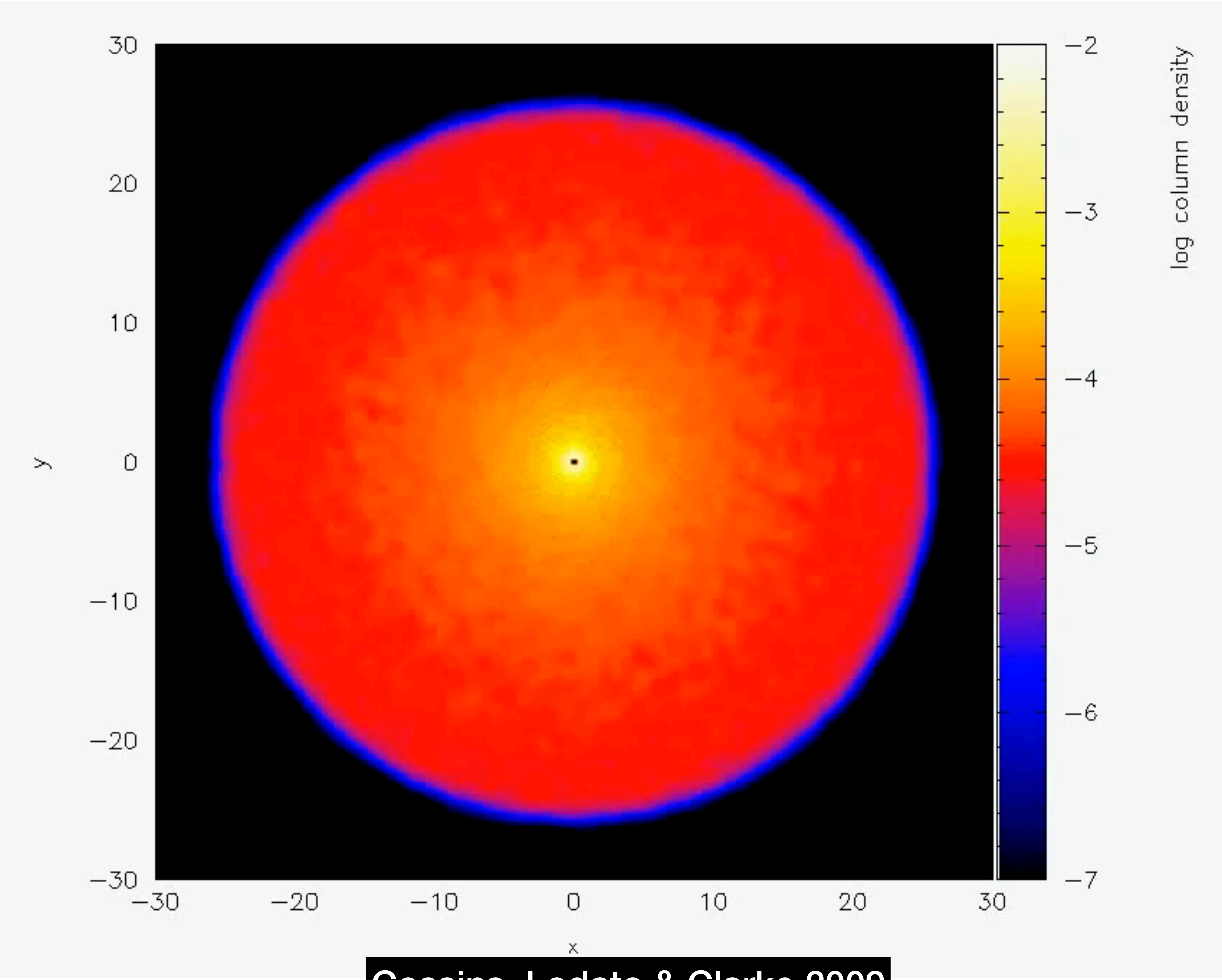
Gravitational instability





Cossins, Lodato & Clarke 2009

Gravitational instability



Cossins, Lodato & Clarke 2009

Exercise for Lecture 4

- Consider a disc with small but finite thickness H<<R
- How do we modify the Lin-Shu dispersion relation? Is thickness a stablizing or destabilizing effect?

$$(\omega - m\Omega)^2 = c_s^2 k^2 - 2\pi G \Sigma |k| + \kappa^2$$